The log-Brunn-Minkowski inequality and its local version

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Outline

The Brunn-Minkowski inequality

$K, L \subset \mathbb{R}^n$ convex bodies

$$\mathrm{vol}(K+L)^{\frac{1}{n}} \geq \mathrm{vol}(K)^{\frac{1}{n}} + \mathrm{vol}(L)^{\frac{1}{n}}$$

The Brunn-Minkowski inequality

$K, L \subset \mathbb{R}^n$ convex bodies, $\lambda \in [0, 1]$

$$\mathrm{vol}((1-\lambda)K + \lambda L)^{\frac{1}{n}} \geq (1-\lambda)\,\mathrm{vol}(K)^{\frac{1}{n}} + \lambda\,\mathrm{vol}(L)^{\frac{1}{n}}$$

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with equality iff K, L are homothetic.

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... if $p \geq 1$.

The conjectured L^p-BM inequality (BLYZ, 2012)

$K, L \subset \mathbb{R}^n$ *centrally symmetric* convex bodies, $\lambda \in [0,1]$, $p \in (0,1]$

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$h_{(1-\lambda)K+_p\lambda L} \neq ((1-\lambda)h_K(u)^p + \lambda h_L(u)^p)^{\frac{1}{p}}$ in general!

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Again, $h_{(1-\lambda)K+_o\lambda L} \neq h_K^{1-\lambda} h_L^{\lambda}$ in general.

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Conjecture

For all c.s. convex bodies K, L and diagonal matrices Λ, the function

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Saroglou:

- Log-BM in dimension $n \Rightarrow$ Weak (B)-conjecture in dimension n.
- (B)-conjecture in all dimensions for $K = B_\infty^n \Rightarrow$ Log-BM in all dimensions.

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Nayar-Tkocz: (B)-conjecture holds for $K = B_1^n, B_2^n$.

The Minkowski and log-Minkowski inequalities

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$\frac{\mathrm{vol}(K)}{n} \log \frac{\mathrm{vol}(L)}{\mathrm{vol}(K)} \le \int h_K \log \frac{h_L}{h_K} \, dS_K$, equality iff K, L are similar.

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Follows from computing the derivative of $\mathrm{vol}((1-\lambda)K +_o \lambda L)^{\frac{1}{n}}$ at 0 and log-BM.

Minkowski uniqueness

The Minkowski inequality (1)

$K, L \subset \mathbb{R}^n$ convex bodies $\Rightarrow \mathrm{vol}(K)^{\frac{n-1}{n}} \mathrm{vol}(L)^{\frac{1}{n}} \leq \frac{1}{n} \int h_L \, dS_K$, with equality iff K, L are homothetic.

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(1) $\Rightarrow$ (2): Multiply $dS_K = dS_L$ by h_K and integrate; use (1) to show that that $\mathrm{vol}(K) \leq \mathrm{vol}(L)$. By the same argument, $\mathrm{vol}(L) \leq \mathrm{vol}(K)$, so $\mathrm{vol}(K) = \mathrm{vol}(L) = \frac{1}{n} \int h_L \, dS_K$.

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(2) $\Rightarrow$ (1): For given K, let K_0 minimize the functional $f(h_L) = \mathrm{vol}(L)^{-\frac{1}{n}} \int h_L \, dS_K$; wlog $\mathrm{vol}(K_0) = \mathrm{vol}(K)$. We claim K_0 is homothetic to K.

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For any $\varphi \in C(S^{n-1})$, let $g(t) = f(h_{K_0} + t\varphi)$; we must have $g'(0) = 0$, giving $\int \varphi \, dS_{K_0} = \int \varphi \, dS_K \Rightarrow S_K = S_{K_0}$. Now use (2).

Log-Minkowski uniqueness

The conjectured log-Minkowski inequality

K, L c.s. convex bodies $\Rightarrow \frac{1}{n}\frac{\mathrm{vol}(L)}{\mathrm{vol}(K)} \leq \int h_K \log \frac{h_L}{h_K}\, dS_K$ with equality iff K, L are *similar*: that is, there exist c.s. convex bodies $K_1, \ldots, K_m$, $\alpha_i > 0$, $T \in GL_n$ such that $K = T(K_1 \times \cdots \times K_m)$ and $L = T(\alpha_1 K_1 \times \cdots \alpha_m K_m)$.

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There are also L^p-Minkowski inequalities and corresponding p-Minkowski uniqueness statements for all $p \in (0, 1]$.

Differentiating the log-BM inequality

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Studied by Colesanti-Livshyts-Marsiglietti and by Kolesnikov-Milman in the class of smooth and strongly convex bodies, and by P. in the class of strongly isomorphic polytopes.

Local log-BM and local p-BM

Theorem (Colesanti-Livshyts-Marsiglietti)

Set $K = B_2^n$. Then for $L \in \mathcal{K}^n_{+,e}$ close enough to K, the log-Brunn-Minkowski inequality holds for K, L.

Write $h_L = e^{\varphi}$, $K_\lambda = (1-\lambda)K +_o \lambda L$. Then $h_{K_\lambda} = e^{\lambda\varphi}$ for all $\lambda \in [0,1]$. Substitute in

$$\mathrm{vol}(K_\lambda) = \frac{1}{n}\int_{S^{n-1}} h_{K_\lambda}\, dS_{K_\lambda} = \frac{1}{n}\int_{S^{n-1}} h_{K_\lambda} \det[D^2 h_{K_\lambda}]\, d\sigma.$$

Using some linear algebra, we compute that $n^2\kappa_n \log \mathrm{vol}(K_\lambda)''(0)$ equals

$$n\int \varphi^2\, d\sigma - \int_{S^{n-1}} |\nabla\varphi|^2\, d\sigma - \frac{1}{\kappa_n}\left(\int_{S^{n-1}} \varphi\, d\sigma\right)^2$$

Decomposing into spherical harmonics shows this is nonpositive!

Local L^p-Brunn-Minkowski

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They showed that $(\mathrm{vol}(K_\lambda)^{\frac{p}{n}})''(0) \leq 0$ implies that for all $\varphi \in C^2(S^{n-1})$,

$$\frac{n-1}{n-p} V(\varphi h_K[2], K[n-2]) + \frac{1-p}{n-p} V(\varphi^2 h_K[1], K[n-1]) - \frac{V(\varphi h_K[1], K[n-1])^2}{\mathrm{vol}(K)} \leq 0$$

Using spectral methods, they proved this inequality for $p \in [p_0(n), 1]$.

Why local-to-global is nontrivial

Whenever $h_{K_\lambda} = (1-\lambda)h_K +_p \lambda h_L$ on some neighborhood, $\mathrm{vol}(K_\lambda)^{\frac{p}{n}}$ is twice differentiable with second derivative given by the LHS of the local L^p-BM inequality.

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It turns out that the first derivative of $\mathrm{vol}(K_\lambda)$ can be computed despite this problem (Alexandrov's lemma), but the second derivative seems out of reach in general.

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Yields equivalence of local and global L^p-BM in general, and in particular proves L^p-BM for $p \in [p_0(n), 1]$.

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Local-to-global - some details (1)

Lemma

Let K and L be polytopes with facet normals $u_i \in S^{n-1}$ and support numbers $h_K(u_i) = \alpha_i$, $h_L(u_i) = \alpha_i e^{s_i}$. Then for any $\lambda \in [0,1]$, $K_\lambda = \{x : \langle x, u_i \rangle \leq \alpha_i e^{\lambda s_i}\}$.

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Let K and L be polytopes with facet normals $u_i \in S^{n-1}$ and support numbers $h_K(u_i) = \alpha_i$, $h_L(u_i) = \alpha_i e^{s_i}$. Then for any $\lambda \in [0, 1]$, $K_\lambda = \{x : \langle x, u_i \rangle \leq \alpha_i e^{\lambda s_i}\}$.

Note that K_λ may not have facets corresponding to all the u_i!

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The lemmata enable us to compute $\mathrm{vol}(K_\lambda)'$. But if the K_λ are strongly isomorphic, then each facet of K_λ satisfies the assumptions of the lemma as well, which lets us compute $\log \mathrm{vol}(K_\lambda)''$. The result is the local log-BM formula.

Local-to-global - some details (2)

So assuming the local log-BM inequality, for any neighborhood $U \subset [0,1]$ in which all the $\{K_\lambda : \lambda \in U\}$ are strongly isomorphic, we have $\log \mathrm{vol}(K_\lambda)'' \leq 0$. How do we go from here to a proof that local log-BM implies global log-BM?

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Essentially, this boils down to the question of when K_λ can change its strong isomorphism class. Let's start with an easier question: when does a polytope defined by varying support numbers $h_i(\lambda)$ lose or gain a facet?

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But since the $h_i(\lambda)$ are analytic, any linear combination of them can change sign only at a finite number of points in $[0,1]$. So we obtain that $\log \mathrm{vol}(K_\lambda)'$ is decreasing except at a finite number of points in $[0,1]$, and a continuity argument finishes the proof.

Local Log-BM in dimension 2

For $K, L \in \mathcal{K}_e^2$, define the inradius and circumradius of L w.r.t. K:

$$r(L,K) = \min_{u \in S^{n-1}} \frac{h_L(u)}{h_K(u)} \qquad R(L,K) = \max_{u \in S^{n-1}} \frac{h_L(u)}{h_K(u)}$$

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$$\Rightarrow 2\,\mathrm{vol}(K)\,\mathrm{vol}(L) - 4V(K,L)^2 + \mathrm{vol}(K) \int \frac{h_L^2}{h_K}\, dS_K \leq 0$$

which is precisely local log-BM in dimension 2.

Questions?

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Thank you!