No good dimension reduction in the trace class norm

Gideon Schechtman

BIRS, February 2020

Based on a joint result

with Assaf Naor and Gilles Pisier

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$M = (M, d)$ a metric space. $X = (X, \|\cdot\|)$ a normed space. We say that M embeds into X with distortion C if there is a $f : M \to X$ such that

$$d(x,y) \le \|x - y\| \le Cd(x,y), \quad \text{for all} \quad x, y \in M$$

The best C is denoted by $C_X(M)$.

We are interested in $k_n^C(X)$ - The smallest k such that for all $S \subset X$ with $|S| = n$ there is a subspace $Y \subset X$ of dimension k such that $C_Y(S) \le C$.

For most of this talk think of $C = 2$.

Tight embeddings of metric spaces in normed spaces

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- $X = \ell_2$: Johnson–Lindenstrauss (84): $k_n^2(\ell_2) = O(\log n)$.

(J–S and Larsen –Nelson (2017): $k_n^{1+\epsilon}(\ell_2) \approx \log n/\epsilon^2$, as $\epsilon \to 0$.)

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On the negative side:

• Matoušek (96): For all n and C there is an n-point metric space M such that if M embeds into a normed space Y with distortion C, then dim $Y \geq n^{\alpha/C}$. ($\alpha > 0$ a universal constant). So

$$k_n^C(\ell_\infty) \geq n^{\alpha/C}.$$

(Also, JLS (87): $k_n^C(\ell_\infty) \leq n^{O(1/C)}$.)

• Brinkman–Charikar (2003): For some universal $\alpha > 0$, $k_n^2(\ell_1) \geq n^\alpha$.

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The right hand side bound is due to Andoni,Naor,Neiman (2017).)

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The trace class

The purpose of our result and this lecture is to add one more such example: The trace class (AKA Schatten–von-Neumann 1, Nuclear norm).

Given a linear operator $T : \ell_2 \to \ell_2$ define

$$\|T\|_{S_p} = (\mathrm{trace}(T^*T)^{p/2})^{1/p} = (\sum(\sigma_i(T))^p)^{1/p}$$

where $\sigma_i(T)$ are the singular values of T.

$$\|T\|_{S_\infty} = \max \sigma_i(T) = \text{operator norm},$$

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The main result

Theorem

(Naor, Pisier, S. Just appeared online in DCG)

$$k_n^C(S_1) \geq n^{\alpha/C^2}.$$

($\alpha > 0$ universal.)

Meaning: For all n there are n points in S_1 such that if Y is a subspace of S_1 of dimension k into which these n points embed with distortion C then $k \geq n^{\alpha/C^2}$.
Note that ℓ_1 embeds with distortion 1 into S_1 (as the set of diagonal matrices). The bad sets we use are the same as those used by Brinkman and Charikar - the diamond graphs. (So our theorem is a strengthening of the Brinkman–Charikar result.)

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Diamond

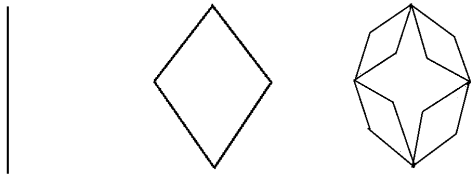

Figure: Diamonds D_0, D_1, D_2

Strategy of proof

The proof imitates a geometrical proof of the Brinkman–Charikar theorem (due essentially to Lee and Naor (2004)). It consists of two stages:

• D_n “doesn’t well embed" in S_p for $p > 1$. (With some precise quantitative estimates).

• A k dimensional subspace of S_1 is close to a natural subspace of S_p and in particular "well embeds" in S_p. (Again with a precise quantitative estimate).

The proof of the first • is very similar to a the one for ℓ_1 and uses the estimates of the uniform convexity modulus of S_p, $1 < p < 2$ (which are the same as for ℓ_p, $1 < p < 2$).

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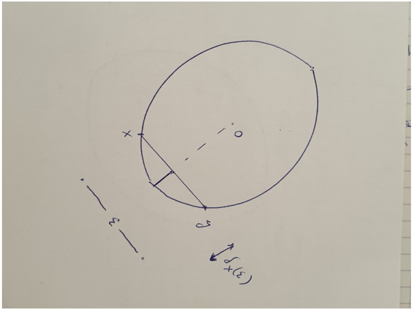
uniform convexity

The uniform convexity modulus of a normed space X is the function

$$\delta_X(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| \; ; \;\; \|x\|, \|y\| \leq 1, \;\; \|x-y\| \geq \epsilon\}.$$



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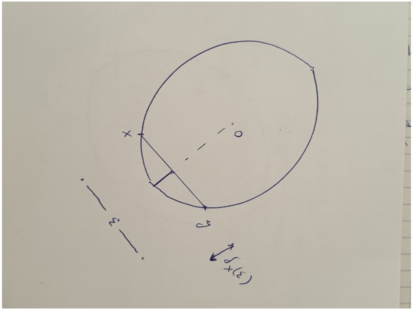

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Lemma

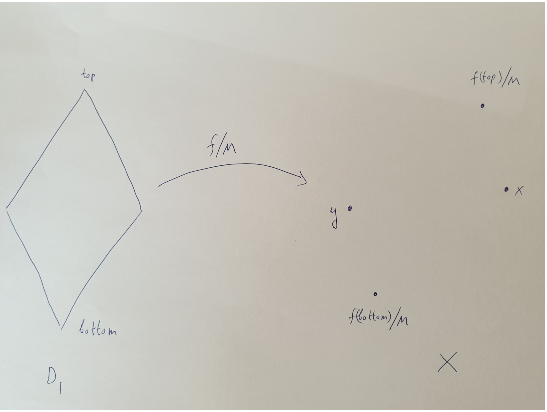
$f : D_1 :\to X$,

$$d(x,y) \leq \|f(x) - f(y)\| \leq Md(x,y).$$

Then,

$$2 \leq \|f(top) - f(bottom)\| \leq 2M(1 - \delta(2/M)).$$

Proof:



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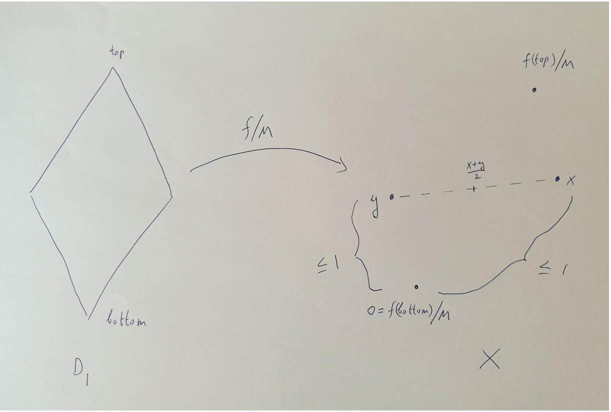

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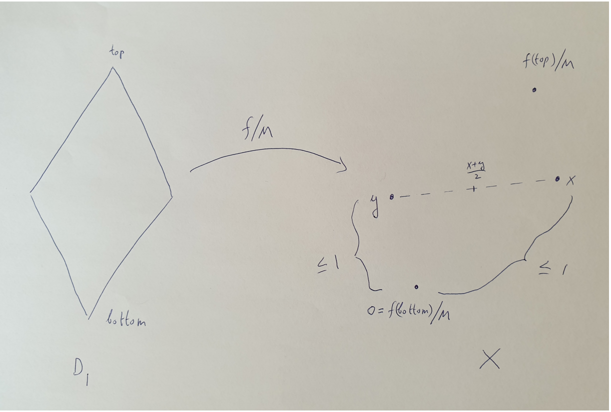

$\|x - y\| \geq 2/M$ so $\|\frac{x+y}{2}\| \leq 1 - \delta(2/M)$.

Similarly, $\|\frac{x+y}{2} - f(top)/M\| \leq 1 - \delta(2/M)$

so $\|f(bottom)/M - f(top)/M\| \leq 2(1 - \delta(2/M))$.

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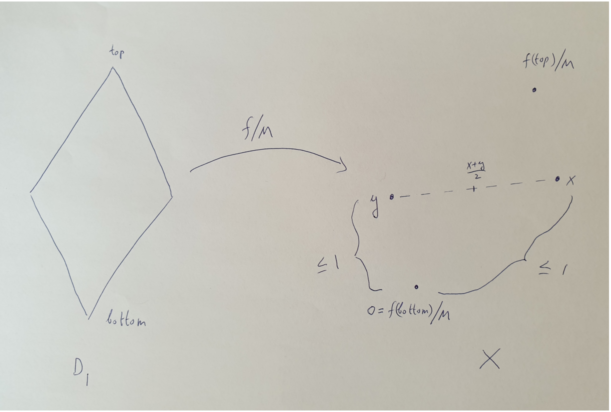

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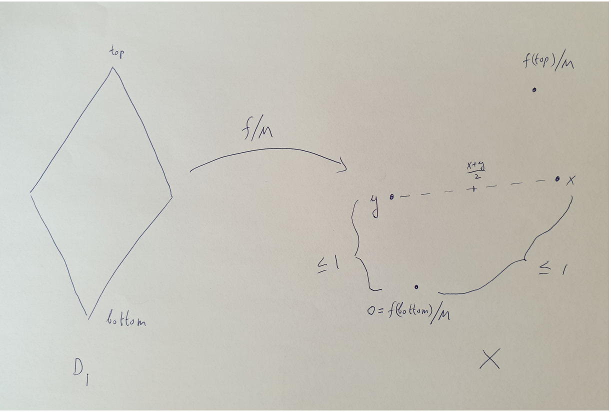

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non embedding of D_n in S_p

Corollary

Let M_n be the least M such that there is $f : D_n \to X$ with

$$d(x,y) \le \|f(x) - f(y)\| \le M d(x,y).$$

Then

$$M_{n-1} \le M_n(1 - \delta_X(2/M_n)).$$

From this one gets a lower bound on M_n in terms of δ_X.

$$\delta_{\ell_p}(\epsilon),\ \delta_{S_p}(\epsilon) \ge c(p-1)\epsilon^2, \quad 1 < p \le 2.$$

From this one gets, for $X = \ell_p,\ S_p$

$$M_n \ge (c(p-1)n)^{1/2}.$$

Which is what we meant by "D_n doesn't well embed in S_p".

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We now deal with the second •:

• A k-dimensional subspace of S_1 (resp. ℓ_1) "well embeds" in S_p (resp. ℓ_p).

Here there is a difference between the cases of ℓ_p and S_p. For ℓ_p a k-dimensional subspace of ℓ_1 embeds in $\ell_1^{\bar{k}}$ with $\bar{k}$ almost linear in k (polynomial dependence is enough for us), and thus embeds with distortion $\bar{k}^{1-\frac{1}{p}}$ in $\ell_p^{\bar{k}}$.

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Problem:

Given k what is the order of the smallest m such that every k-dimensional subspace of S_1 2-embeds into S_1^m?

No polynomial bound is known. I conjecture that there is no such bound. Some weak indication is in a recent result of Regev and Vidick:

[RV]:

For some universal constant $c > 0$ and for all k there are $A_1, \dots, A_k$ in S_1^m (with $m = 2^{k/2}$) such that if $\{A_1, \dots, A_k\}$ embed in S_1^d with distortion $1 + \frac{1}{k^c}$ then $d \geq m/2$.

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No polynomial bound is known. I conjecture that there is no such bound. Some weak indication is in a recent result of Regev and Vidick:

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For some universal constant $c > 0$ and for all k there are $A_1, \ldots, A_k$ in S_1^m (with $m = 2^{k/2}$) such that if $\{A_1, \ldots, A_k\}$ embed in S_1^d with distortion $1 + \frac{1}{k^c}$ then $d \geq m/2$.

embedding subspaces of S_1 in S_p

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Theorem

For each k and $1 < p \le 2$, a k-dimensional subspace X of S_1 embeds with distortion $k^{1-\frac{1}{p}}$ into S_p. i.e., $C_{S_p}(X) \le k^{1-\frac{1}{p}}$.

The main tool is a

Non-commutative Lewis' lemma:

Let X be a k-dimensional subspace of S_1. Then it admits a basis $T_1, \ldots, T_k$ satisfying

$$\operatorname{trace}\left[\frac{1}{2}(T_i^* T_j + T_j^* T_i)M^{-1/2}\right] = \delta_{i,j}, \text{ for all } i, j \in \{1, \ldots, k\}.$$

$M = \sum_s T_s^* T_s$.

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The (commutative) Lemma of Dan Lewis (70-s) says that

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If X is a k-dimensional subspace of $L_p(0,1)$ (or ℓ_p), then it admits a basis $f_1, \dots, f_k$ satisfying

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This means that X is isometric to a subspace $\bar{X}$ of an $L_1(\mu)$ for some probabiity μ, and $\bar{X}$ admits an orthonormal basis $\{g_i\}$ with $\sum_i g_i^2 \equiv k$. Then the identity map between $\bar{X}$ with the $L_1(\mu)$ norm and $\bar{X}$ with the $L_p(\mu)$ norm shows that $C_{L_p}(X) \le k^{1-1/p}$.

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In S_1 the situation is a bit different. The problem is that there is no proper "change of density": $\mathrm{trace}(TM^{1/2})$ is not a norm isometric to the S_1 norm.

It turns out however that the map $T \to TM^{\frac{p-1}{2p}}$ gives a good embedding of X into S_p.

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Stronger theorem

One can also use the less intuitive notion of “Markov convexity" instead of uniform convexity and get a bit more:

“Improved Theorem"

For each n there is a set of n points in S_1 (even ℓ_1) which are “quotient of a subset" of a subspace X of S_1 with distortion C only if $\dim(X) = n^{\alpha/C^2}$. ($\alpha > 0$ universal.)

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