The convex hull of random points on the boundary of a simple polytope

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Notions and Definitions

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We are interested in

- the expected number of vertices $\mathbb{E} f_0(K_N)$,
- the expected number of facets $\mathbb{E} f_{n-1}(K_N)$,
- the expectation of the volume difference

$$\mathrm{vol}_n(K) - \mathbb{E}\,\mathrm{vol}_n(K_N)$$

  of K and K_N.

Notions and Definitions

Since explicit results for fixed N cannot be expected we investigate the asymptotics as $N \to \infty$.

Introduction

For all convex bodies K in $\mathbb{R}^n$

$$c(n) \lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(K,N)}{\left(\frac{vol_n(K)}{N}\right)^{\frac{2}{n+1}}} = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu(x)$$

where $\kappa(x)$ denotes the generalized Gauß-Kronecker curvature.

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For a polytope P the formula gives 0, since the curvature of a polytope is 0 almost everywhere.

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- The integral

$$\int_{\partial K} \kappa(x)^{\frac{1}{n+1}} d\mu(x)$$

is called affine surface area.

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is called a flag of P. The number of all flags of P is denoted by

$$\mathrm{flag}(P).$$

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We have for polytopes P in $\mathbb{R}^n$

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(P) - \mathbb{E}(P,N)}{\frac{1}{N}(\ln N)^{n-1}} = \frac{\mathrm{flag}(P)\,\mathrm{vol}_n(P)}{(n+1)^{n-1}(n-1)!}.$$

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- This formula was shown by Barany and Buchta in arbitrary dimension.
- The phenomenon that flag(P) shows up in such formulae were first shown for the floating body by S..

Introduction

Barany and Larman: Let K be a convex body. Then there is N_0 such that for all $N \geq N_0$

$$\begin{aligned} c_1 \left(\mathrm{vol}_n(K) - \mathrm{vol}_n(K_{\frac{1}{N}\mathrm{vol}_n(K)}) \right) &\leq \mathrm{vol}_n(K) - \mathbb{E}(K,N) \\ &\leq c_2 \left(\mathrm{vol}_n(K) - \mathrm{vol}_n(K_{\frac{1}{N}\mathrm{vol}_n(K)}) \right). \end{aligned}$$

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Let K be a convex body in $\mathbb{R}^n$ and let $f : \partial K \to \mathbb{R}_+$ be a continuous, positive function with $\int_{\partial K} f(x) d\mu_{\partial K}(x) = 1$ where $\mu_{\partial K}$ is the surface measure on ∂K. Let $\mathbb{P}_f$ be the probability measure on ∂K given by $\mathbb{P}_f(x) = f(x) d\mu_{\partial K}(x)$.

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Let κ be the (generalized) Gauß-Kronecker curvature and $\mathbb{E}(f, N)$ the expected volume of the convex hull of N points chosen randomly on ∂K with respect to $\mathbb{P}_f$. Then, under some regularity conditions on the boundary of K

$$\lim_{N\to\infty} \frac{\mathrm{vol}_n(K) - \mathbb{E}(f, N)}{\left(\frac{1}{N}\right)^{\frac{2}{n-1}}} = c_n \int_{\partial K} \frac{\kappa(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}} d\mu_{\partial K}(x),$$

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- This formula was shown by S. and Werner for convex bodies in which a small Euclidean ball rolls freely and which rolls freely in a big Euclidean ball.
- At the same time Reitzner showed this formula for convex bodies with C^2_+-boundary.

Introduction

The general results for the number of ℓ-dimensional faces $f_\ell(K_N)$ are due to Wieacker, Bárány and Buchta, and Reitzner : if K is a smooth convex body and $\ell \in \{0, \dots, n-1\}$, then

$$\mathbb{E} f_\ell(K_N) = c(n,\ell)\,\mathrm{as}(K)\,N^{\frac{n-1}{n+1}}(1+o(1)), \tag{1}$$

and if P is a polytope, then

$$\mathbb{E} f_\ell(P_N) = c(n,\ell)\,\mathrm{flag}(P)\,(\ln N)^{n-1}(1+o(1)). \tag{2}$$

Our main results

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Theorem (Reitzner, S. and Werner)

Choose N random points uniformly on the boundary of a simple polytope P. For the expected number of facets of the random polytope P_N, we have

$$\mathbb{E}(f_{n-1}(P_N)) = c_{n,n-1} f_0(P))(\ln N)^{n-2}(1 + O((\ln N)^{-1}),$$

with some $c_{n,n-1} > 0$.

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with some $c_{n,n-1} > 0$.

Our proof shows that the crucial contribution to the number of faces of P_N comes from those faces that are not contained in the boundary of P and whose vertices are from exactly two facets of P.

Our main results

Theorem (Reitzner, S. and Werner)

For the expected volume difference between a simple polytope $P \subset \mathbb{R}^n$ and the random polytope P_N with vertices chosen from the boundary of P, we have

$$\mathrm{vol}_n(P) - \mathbb{E}\,\mathrm{vol}_n(P_N) = c_{n,P} N^{-\frac{n}{n-1}} \left(1 + O(N^{-\frac{1}{(n-1)(n-2)}})\right)$$

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We expect for arbitrary polytopes P

$$\mathrm{vol}_n(P) - \mathbb{E}\,\mathrm{vol}_n(P_N)) = c_n \frac{\mathrm{flag}(P)\,\mathrm{vol}_n(P)}{N^{\frac{n}{n-1}}} (1 + O(N^{-\frac{1}{(n-1)(n-2)}}))$$

Proof.

Those facets of P_N that are contained in a facet of P do not contribute to the difference volume

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All sets $[v, F]$ are contained in $P \setminus P_N$ and their pairwise intersections are nullsets.

We put

$$A_N = \bigcup_{v \in \mathcal{F}_0(P)} \bigcup_{\substack{N_F \in \mathcal{N}(v,P) \\ F \not\subseteq \partial P}} [F, v] \qquad \text{and} \qquad D_N = P \setminus (P_N \cup A_N) \qquad (3)$$

where D_N is the subset of $P \setminus P_N$ not covered by one of the simplices $[F, v]$.

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where D_N is the subset of $P \setminus P_N$ not covered by one of the simplices $[F, v]$. It follows

$$\mathrm{vol}_n(A_N) = \sum_{F \not\subseteq \partial P} \mathrm{vol}_n([v(F), F]) = \frac{1}{n} \sum_{F \not\subseteq \partial P} \mathrm{vol}_{n-1}(F) d(F, v(F))$$

where $d(F, v(F))$ is the distance of the vertex $v(F)$ to the hyperplane spanned by F.

Let $P_N = [x_1, \ldots, x_N]$. Then

$$\mathrm{vol}_n(A_N) = \frac{1}{n} \sum_{i_1,\ldots,i_n=1}^{N} \mathrm{vol}_{n-1}([x_{i_1}, \ldots, x_{i_n}]) d([x_{i_1}, \ldots, x_{i_n}], v([x_{i_1}, \ldots, x_{i_n}])) \chi([x_{i_1}, \ldots, x_{i_n}] \text{ is a facet of } P_N) \chi([x_{i_1}, \ldots, x_{i_n}] \not\subseteq \partial P)$$

where $v([x_{i_1}, \ldots, x_{i_n}])$ is the vertex whose normal cone contains the normal to the hyperplane spanned by $x_1, \ldots, x_n$.

Let $P_N = [x_1, \dots, x_N]$. Then

$$\begin{aligned}\mathrm{vol}_n(A_N) &= \frac{1}{n} \sum_{i_1,\dots,i_n=1}^{N} \mathrm{vol}_{n-1}([x_{i_1},\dots,x_{i_n}]) d([x_{i_1},\dots,x_{i_n}], v([x_{i_1},\dots,x_{i_n}])) \\ &\qquad \chi([x_{i_1},\dots,x_{i_n}] \text{ is a facet of } P_N)\chi([x_{i_1},\dots,x_{i_n}] \not\subseteq \partial P)\end{aligned}$$

where $v([x_{i_1},\dots,x_{i_n}])$ is the vertex whose normal cone contains the normal to the hyperplane spanned by $x_1,\dots,x_n$. Therefore

$$\begin{aligned}&\mathrm{vol}_n(P) - \mathrm{vol}_n(P_N) = \mathrm{vol}_n(A_N) + \mathrm{vol}_n(D_N) \\ = &\frac{1}{n} \sum_{i_1,\dots,i_n=1}^{N} \mathrm{vol}_{n-1}([x_{i_1},\dots,x_{i_n}]) d([x_{i_1},\dots,x_{i_n}], v([x_{i_1},\dots,x_{i_n}])) \\ &\quad \chi([x_{i_1},\dots,x_{i_n}] \text{ is a facet of } P_N)\chi([x_{i_1},\dots,x_{i_n}] \not\subseteq \partial P) \ + \mathrm{vol}_n(D_N).\end{aligned}$$

Since $\text{vol}_{n-1}(\partial P) = 1$

$$\mathbb{E}(\text{vol}_n(P) - \text{vol}_n(P_N))$$

$$= \frac{1}{n} \int_{\partial P} \cdots \int_{\partial P} \sum_{i_1,\dots,i_n=1}^{N} \text{vol}_{n-1}([x_{i_1},\dots,x_{i_n}]) d([x_{i_1},\dots,x_{i_n}], v([x_{i_1},\dots,x_{i_n}]))$$

$$\chi([x_{i_1},\dots,x_{i_n}] \text{ is a facet of } P_N) \chi([x_{i_1},\dots,x_{i_n}] \not\subseteq \partial P) \; + \text{vol}_n(D_N) dx_1 \cdots dx_N$$

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= &\frac{1}{n}\binom{N}{n}\int_{\partial P}\cdots\int_{\partial P}\mathrm{vol}_{n-1}([x_1,\dots,x_n])d([x_1,\dots,x_n],v([x_1,\dots,x_n])) \\
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\end{aligned}
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We have

$$\begin{aligned}&\int_{\partial P}\cdots\int_{\partial P}\chi([x_1,\dots,x_n]\text{ is a facet of }P_N)dx_{n+1}\cdots dx_N\\&=\mathbb{P}([x_1,\dots,x_n]\text{ is a facet of }P_N)\\&=\mathrm{vol}_{n-1}(\partial P\cap H^-)^{N-n}+\mathrm{vol}_{n-1}(\partial P\cap H^+)^{N-n}\end{aligned}$$

where H is the hyperplane spanned by $x_1,\dots,x_n$.

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We are choosing H^- to be the halfspace with H^- is the halfspace with $\mathrm{vol}_n(P\cap H^-)\geq \mathrm{vol}_n(P\cap H^+)$.

Therefore

$$\begin{aligned}
&\mathbb{E}(\mathrm{vol}_n(P) - \mathrm{vol}_n(P_N)) \\
= &\frac{1}{n}\binom{N}{n} \int_{\partial P} \cdots \int_{\partial P} (\mathrm{vol}_{n-1}(\partial P \cap H^-(x_1, \dots, x_n))^{N-n} \\
&+ \mathrm{vol}_{n-1}(\partial P \cap H^+(x_1, \dots, x_n))^{N-n}) \,\mathrm{vol}_{n-1}([x_1, \dots, x_n]) \\
&\ d([x_1, \dots, x_n], v(x_1, \dots, x_n))\chi([x_1, \dots, x_n] \not\subseteq \partial P) dx_1 \cdots dx_n + O(2^{-N}) \\
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where $v(x_1,\dots,x_n)$ is the vertex such that the normal to the hyperplane spanned by $x_1,\dots,x_n$ is an element of the normal cone of v.

Lemma (Zähle)

Let ∂K be a rectifiable manifold and let $g(x_1, \ldots, x_n)$ be a continuous function. Then there is a constant β such that

$$\int_{\partial K} \cdots \int_{\partial K} \chi(x_1, \ldots, x_n \text{ in general position}) g(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

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$$= \frac{(n-1)!}{\beta}\int_{S^{n-1}}\int_{\mathbb{R}}\int_{\partial K\cap H}\cdots\int_{\partial K\cap H} g(x_1,\dots,x_n)$$

$$\chi_{n-1}([x_1,\dots,x_n])\prod_{j=1}^{n}J(T_{x_j},H)^{-1}\,dx_1\cdots dx_n\,dh\,du$$

with dx, du, dh denoting integration with respect to the Hausdorff measure on the respective range of integration, and $J(T_{x_j},H)$ is the sine of the angle between H and T_{x_j}.

Now we apply the Lemma of Zähle

$$
\begin{aligned}
&\mathbb{E}(\mathrm{vol}_n(P) - \mathrm{vol}_n(P_N)) \\
= &\frac{1}{n}\binom{N}{n}\frac{(n-1)!}{\beta}\int_{S^{n-1}}\int_{\mathbb{R}}\int_{\partial P\cap H(u,h)}\cdots\int_{\partial P\cap H(u,h)}(\mathrm{vol}_{n-1}(\partial P\cap H^{-}(u,h))^{N-n} \\
&\mathrm{vol}_{n-1}([x_1,\dots,x_n])^2 d(H(u,h),v(u))\prod_{j=1}^{n}J(T_{x_j},H(u,h))^{-1} \\
&\chi([x_1,\dots,x_n]\not\subseteq\partial P)dx_1\cdots dx_n dh du \\
&+\int_{\partial P}\cdots\int_{\partial P}\mathrm{vol}_n(D_N)dx_1\cdots dx_N + O(2^{-N}),
\end{aligned}
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where $v(u)$ is the unique vertex with $u\in\mathcal{N}(v(u),P)$.

Up to nullsets the normal cones $\mathcal{N}(v, P)$, $v \in \mathcal{F}_0(P)$ are disjoint

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\begin{aligned}
&\mathbb{E}(\mathrm{vol}_n(P) - \mathrm{vol}_n(P_N)) \\
&= \frac{1}{n}\binom{N}{n}\frac{(n-1)!}{\beta}\sum_{v\in\mathcal{F}_0(P)}\int_{S^{n-1}\cap-\mathcal{N}(v,P)}\int_{\mathbb{R}}\int_{\partial P\cap H(u,h)}\cdots\int_{\partial P\cap H(u,h)} \\
&\quad \mathrm{vol}_{n-1}(\partial P\cap H^-(u,h))^{N-n}\,\mathrm{vol}_{n-1}([x_1,\dots,x_n])^2 d(H(u,h),v) \\
&\quad \prod_{j=1}^{n} J(T_{x_j}, H(u,h))^{-1}\chi([x_1,\dots,x_n]\not\subseteq\partial P)dx_1\cdots dx_n dh du \\
&\quad + \int_{\partial P}\cdots\int_{\partial P}\mathrm{vol}_n(D_N)dx_1\cdots dx_N + O(2^{-N}).
\end{aligned}
$$

Now we remove the assumption $\mathrm{vol}_{n-1}(\partial P) = 1$.

Then we get

$$\frac{1}{n}\binom{N}{n}\frac{(n-1)!}{\beta}\,\mathrm{vol}_{n-1}(\partial P)^{-2-\frac{n(n-2)}{n-1}}$$

$$\sum_{w\in\mathcal{F}_0(P)}\int_{S^{n-1}\cap-\mathcal{N}(w,P)}\int_{\mathbb{R}}\int_{\partial P\cap H(u,h)}\cdots\int_{\partial P\cap H(u,h)}$$

$$\left(\frac{\mathrm{vol}_{n-1}(\partial P\cap H^{-}(u,h))}{\mathrm{vol}_{n-1}(\partial P)}\right)^{N-n}\mathrm{vol}_{n-1}([y_1,\dots,y_n])^2 d(H(u,k),w)$$

$$\prod_{j=1}^{n}J(T_{x_j},H(u,h))^{-1}\chi([y_1,\dots,y_n]\not\subseteq\partial P)dy_1\cdots dy_n dk du$$

$$+\int_{\partial P}\cdots\int_{\partial P}\mathrm{vol}_n(D_N)dy_1\cdots dy_N+O(2^{-N}).$$