Lower deviation estimates in normed spaces

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joint work with Grigoris Paouris and Konstantin Tikhomirov

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Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$; let G be a Gaussian vector. To provide (dimensional) upper bounds for

$$\mathbb{P}(\|G\| \leq \delta \mathbb{E}\|G\|), \quad \delta \in (0,1).$$

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- Discover the probabilistic principles to be exploited for obtaining finer estimates.

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- (Klartag, Vershynin, '04) Linked small ball estimates with the one-sided randomized Dvoretzky theorem; emphasized the role of the parameter

$$0 < \delta < 1, \quad d(\delta) = d(\|G\|, \delta) := -\log \mathbb{P}(\|G\| \leq \delta \mathrm{med}(\|G\|)).$$

If $0 < \delta < 1$, then

$$\mathbb{P}(\|G\| \leq \varepsilon \mathrm{med}(\|G\|)) \leq \varepsilon^{\frac{d(\delta)-1/2}{\log(1/\delta)}}, \quad 0 < \varepsilon < \delta.$$

Tools used:

- (Cordéro-Erasquin, Fradelizi, Maurey, 04). *B-inequality for the Gaussian measure*. The following map is concave

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- *The ℓ_∞^n-norm.* Note that $k(\|G\|_\infty) \asymp \log n$. If we use $\mathrm{Var}[\|G\|_\infty] \asymp (\log n)^{-1}$ we obtain $d(\|G\|_\infty, 1/2) \asymp (\log n)^2$. However, one has (by direct calculations)

$$\mathbb{P}(\|G\|_\infty \leq \delta\mathbb{E}\|G\|_\infty) \leq \exp(-cn^{1-c\delta^2}), \quad 0 < \delta < 1/2.$$

In particular, $d(\|G\|_\infty, \delta) \geq cn^{1-c\delta^2}$.

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  What does this say for Gaussian inequalities? There exists a $T \in GL(n)$ such that $d(\|TG\|, 1/2) \geq e^{c\sqrt{\log n}}$.

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- Exploit further the local structure.

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Theorem (Paouris, Tikhomirov, V.)

Let $\|\cdot\|$ be any norm in $\mathbb{R}^n$. We have the following:

- *If $\mathbb{E}|\partial_i\|G\|| = \mathbb{E}|\partial_j\|G\||$ for all $i, j = 1, \ldots, n$, then*

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Theorem (cont'd)

- *In the general case, there exists $T \in GL(n)$ such that*

$$d(\|TG\|,\delta) \geq cn^{1/4-c\delta^2}, \quad 0<\delta<1/2.$$

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    - Apply the variance-sensitive concentration inequality.

Thank you for your attention!