Constraint convex bodies with maximal affine surface area

(based on joint work with O. Giladi, H. Huang and C. Schütt)

BACKGROUND I

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$p \neq -n$,

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Question: Can we get **continuous** affine invariants?

$-\infty \leq p \leq \infty$, $p \neq -n$,

inner and outer maximal affine surface areas

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- $IS_0(K) = os_0(K) = n|K|, \quad IS_n(K) = OS_n(K) = n|B_2^n|$

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Lemma

- *For* $0 \leq p \leq n$*,* $K \to IS_p(K)$ *is continuous in the Hausdorff metric*
- *For* $n \leq p \leq \infty$*,* $K \to OS_p(K)$ *is continuous in the Hausdorff metric*
- *For* $-n \leq p \leq 0$*,* $K \to os_p(K)$ *is continuous in the Hausdorff metric*

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What can be said about K_0?

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Theorem (Baranyi)
Let K be a convex body in $\mathbb{R}^2$. Then there is a unique convex body $K_0 \subset K$ such that

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1. K_0 related to parabolic arcs

Let $\mathcal{P}_N(K)$ be the set of all convex polygones contained in K with vertices from the lattice

$$\frac{1}{N} \quad \mathbb{Z}^2$$

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2. K_0 is the limit shape of lattice polygones

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Theorem (Giladi+Huang+Schütt+W)

There is a constant $c > 0$ such that for all $n \in \mathbb{N}$, all $0 \le p \le n$ and all convex bodies $K \subseteq \mathbb{R}^n$,

$$\frac{1}{n^{5/6}}\left(\frac{c}{L_K}\right)^{\frac{2np}{n+p}}\frac{IS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} \le \frac{IS_p(K)}{|K|^{\frac{n-p}{n+p}}} \le \frac{IS_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}}$$

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Equality holds trivially in the right inequality if $p = 0, n$. If $p \neq 0, n$, equality holds in the right inequality iff K is a centered ellipsoid.

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In particular for $p=1$, $c(n,1)=c\;n^{\frac{1}{n}}$,

$$\frac{c\;n^{\frac{1}{n}}}{n^{5/6}}\;\frac{1}{L_K}\;|K|^{\frac{n-1}{n+1}}\leq\; IS_1(K)=as_1(K_0)\leq\; c\;n^{\frac{1}{n}}\;|K|^{\frac{n-1}{n+1}}$$

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Proof of RHS: L_p **affine isoperimetric inequality**

(Lutwak, Hug, Deping Ye+W)

$$as_p(K) \leq as_p(B_2^n) \frac{|K|^{\frac{n-p}{n+p}}}{|B_2^n|^{\frac{n-p}{n+p}}}$$

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$$IS_p(K) = \max_{C \subset K} as_p(C) \leq \frac{as_p(B_2^n)}{|B_2^n|^{\frac{n-p}{n+p}}} \max_{C \subset K} |C|^{\frac{n-p}{n+p}}$$

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Proof of LHS: We use

Thin Shell Theorem (Paouris; Guédon+E.Milman)
There are constants $0 < c_1 < c_2 < 1$ such that for all convex bodies K in $\mathbb{R}^n$ in isotropic position

$$\left|\{x \in K : c_1 L_K \sqrt{n} \leq \|x\| \leq c_2 L_K \sqrt{n}\}\right| \geq \frac{1}{2}$$