Dual curvature measures and Orlicz-Minkowski problems

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  - Polar body:
$$K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \le 1 \ \text{ for all } \ y \in K\} \in \mathscr{K}^n_{(o)}.$$

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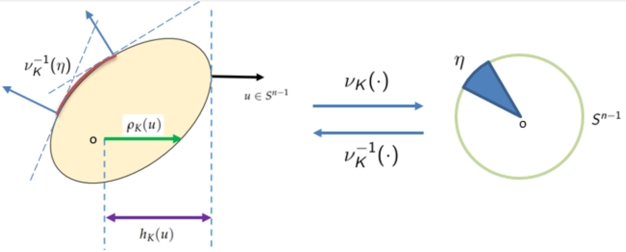
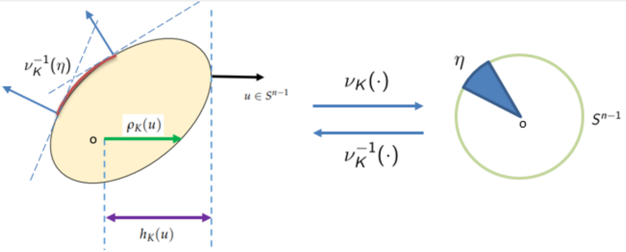


Figure: Support function, radial function and Gauss map



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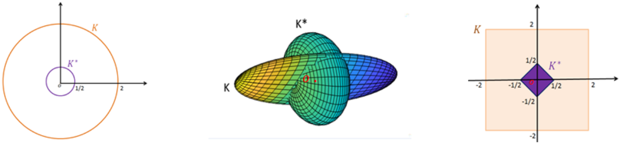

Figure: Polar body

Aleksandrov body

For $f \in C^+(\Omega)$ (positive continuous function on Ω), the Aleksandrov body (Wulff shape) associated with f is

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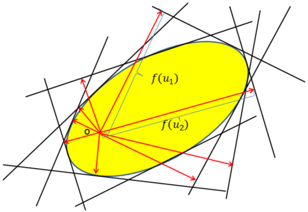

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✧ Volume: $V(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) dS(K, u) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^n du.$

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For a given nonzero finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions on μ such that $\mu = S(K,\cdot)$ for some $K \in \mathscr{K}^n_{(o)}$?

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- Monge-Ampère type equation:

$$f = \det\left(\nabla^2 h + hI\right).$$

Necessary condition

✧ A measure μ is *not concentrated on any closed hemisphere* if

$$\int_{S^{n-1}} \langle u, v \rangle_+ \, d\mu(u) > 0 \quad \text{for any } v \in S^{n-1},$$

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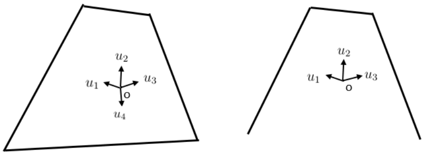

Figure: Support of μ on the plane

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Under what conditions on a finite Borel measure μ and $\phi : (0, \infty) \to (0, \infty)$, does there exist a $K \in \mathscr{K}^n_{(o)}$ such that for some constant $\tau > 0$,

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- Contributions: Haberl-Lutwak-Yang-Zhang, 2010; Huang-He, 2012; Li, 2014; Wu-Xi-Leng, 2018; Sun-Long, 2015; Sun-Zhang, 2018; Sun, 2018, etc.

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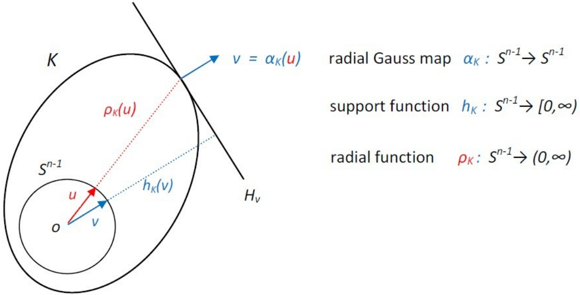

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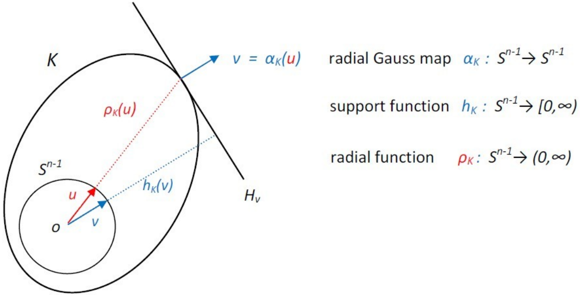

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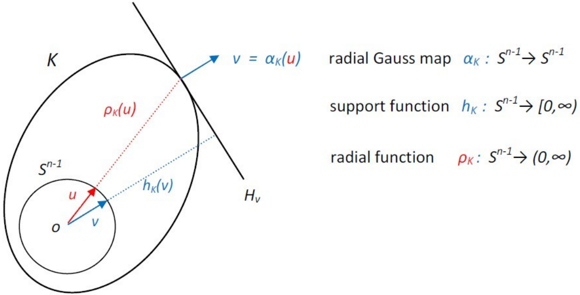

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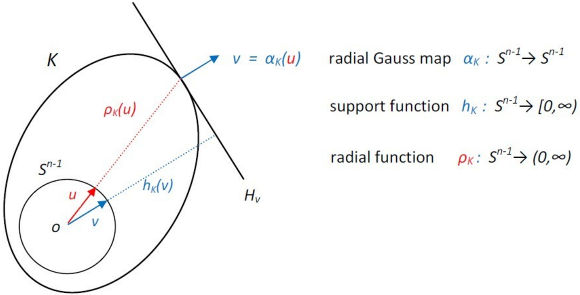

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- $\widetilde{C}_{G,\psi}$ is absolutely continuous to $S(K,\cdot)$, etc.

Uniqueness under certain conditions

Let $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$ be continuous. Suppose that $G_t > 0$ (or $G_t < 0$) on $(0,\infty) \times S^{n-1}$ and that if

$$\frac{G_t(t,u)}{\psi(s)} \geq \frac{\lambda G_t(\lambda t,u)}{\psi(\lambda s)} \quad \left(\text{or} \quad \frac{G_t(t,u)}{\psi(s)} \leq \frac{\lambda G_t(\lambda t,u)}{\psi(\lambda s)}, \quad \text{respectively}\right) \tag{1}$$

for some $\lambda, s, t > 0$ and $u \in S^{n-1}$, then $\lambda \geq 1$. If $K, K' \in \mathscr{K}^n_{(o)}$ are both polytopes or both have support functions in C^2 and

$$\widetilde{C}_{G,\psi}(K,\cdot) = \widetilde{C}_{G,\psi}(K',\cdot),$$

then

$$K = K'.$$

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For which nonzero finite Borel measures μ on S^{n-1} and continuous functions $G : (0,\infty) \times S^{n-1} \to (0,\infty)$ and $\psi : (0,\infty) \to (0,\infty)$, do there exist $\tau \in \mathbb{R}$ and $K \in \mathscr{K}^n_{(o)}$ such that

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- ✧ $tG(t,\cdot)=t^q$, $\psi(t)=t^p$: $\mu=\widetilde{C}_{p,q}(K,\cdot)$? (Lutwak-Yang-Zhang, ADV, 2019)
- ✧ $tG(t,\cdot)=t^n,\psi(t)=\psi(t)$: $\psi(h_K)\mu=S(K,\cdot)$? (Orlicz-Minkowski problem)
- ✧ $tG_t(t,\cdot)=1,\psi(t)=t^p$: $d\mu=\rho_K^p dJ(K,v)$? (L_p Aleksandrov problem)

Monge-Ampère type equation

- ∇h: gradient vector of h, w.r.t. an orthonormal frame on S^{n-1};
- $\nabla^2 h$: Hessian matrix of h w.r.t. an orthonormal frame on S^{n-1};
- ι: the identity map on S^{n-1};
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The corresponding equivalent Monge-Ampère type equation for this general dual Orlicz-Minkowski problem states that for given G, ψ, and $f : S^{n-1} \to [0, \infty)$, an $h : S^{n-1} \to (0, \infty)$ and $\tau \in \mathbb{R}$,

$$f = \frac{\tau h}{\psi \circ h} P(\nabla h + h\iota) \det(\nabla^2 h + hI), \tag{2}$$

where $P(x) = |x|^{1-n} G_t(|x|, \bar{x})$, $\bar{x} = x/|x|$.

Variation formula to derive $\widetilde{C}_{G,\psi}(K, \cdot)$

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Variation formula

$$\lim_{\varepsilon\to0}\frac{\widetilde{V}_G\left([f_\varepsilon]\right)-\widetilde{V}_G(K)}{\varepsilon}=n\int_\Omega g(u)d\widetilde{C}_{G,\psi}(K,u).$$

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Theorem

Under the conditions above, there exists a convex body $K \in \mathscr{K}^n_{(o)}$ such that

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- $\psi = t^p$, $G=t^q$: $p=0, q<0$, our results recover Zhao's result.

Main steps

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✧ **Step 1:** (Condition for G)

For $\{K_i\}_{i=1}^{\infty} \subset \mathscr{K}_{(o)}^n$ satisfying $\widetilde{V}_G(K_i) = |\mu|$, there exists a constant $R > 0$ such that $K_i^* \subset RB^n$.

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  Based on Blaschke selection theorem, there exists a convex body $K_0 \in \mathscr{K}_{(o)}^n$ such that $\widetilde{V}_G(K_0) = |\mu|$ and

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- **Step 3:** (Variation formula)

  The convex body K_0 found in Step 2 is a solution of the dual Orlicz- Minkowski problem, i.e.,

$$\frac{\mu}{\mu(S^{n-1})} = \frac{\widetilde{C}_{G,\psi}(K_0, \cdot)}{\widetilde{C}_{G,\psi}(K_0, S^{n-1})}.$$

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- $\widetilde{C}_{G,\psi}(K,\cdot)$ for $K \in \mathscr{K}_o^n$ satisfies similar properties as $K \in \mathscr{K}_{(o)}^n$.

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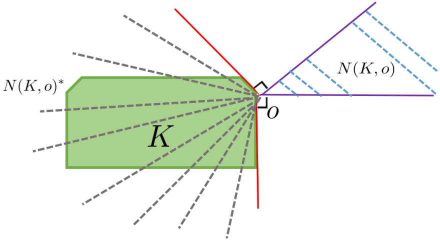

Figure: Normal cone and support cone of a convex body

Reverse radial Gauss image

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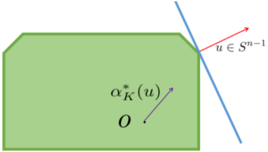

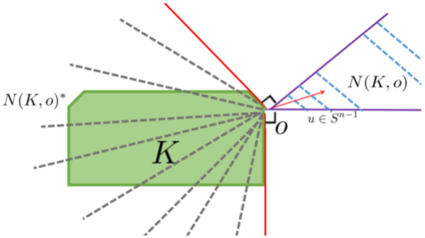

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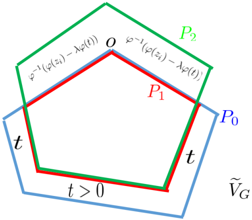
- Multivariable optimization problem: finding $z^0 = (z^0_1, \ldots, z^0_m) \in M$ with

$$M = \left\{ (z_1, \ldots, z_m) \in [0, \infty)^m : \sum_{i=1}^m \lambda_i \varphi(z_i) = \sum_{i=1}^m \lambda_i \varphi(1) \right\}$$

such that $\widetilde{V}_G\left(P\left(z^0\right)\right) = \max\left\{\widetilde{V}_G(P(z)) : \; z \in M\right\}$, where

$$P(z) = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq z_i, \text{ for } i = 1, \ldots, m\}.$$

Contradiction



$P_0 = \cap_{1\le i\le 5}\{x \in \mathbb{R}^2 : \langle x_i, u_i\rangle \le z_i\}$

$\{z_1 = z_2 = 0, z_3, z_4, z_5 > 0.\}$

$\widetilde{V}_G(P_2) > \widetilde{V}_G(P_0)$. A contradiction!

Based on condition of G and φ, we have $P_2 \in M$ and

$$\begin{aligned}\widetilde{V}_G(P_2) &= \widetilde{V}_G(P_2\backslash P_1) + \widetilde{V}_G(P_1)\\ &> \widetilde{V}_G(P_0\backslash P_1) + \widetilde{V}_G(P_1)\\ &= \widetilde{V}_G(P_0) \quad \text{(assumed maximum).}\end{aligned}$$

Main point: Perturbation of height.

- $P_2\backslash P_1$: with height $\varphi^{-1}(\varphi(z_i) - \lambda\varphi(t))$;
- $P_0\backslash P_1$: with height t.

Discrete solution with o in the interior

- $\mu = \sum_{i=1}^{m} \lambda_i \delta_{u_i}$: $\lambda_i > 0$, $i = 1, \ldots, m$, and $\{u_1, \ldots, u_m\} \subset S^{n-1}$ not contained in a closed hemisphere.

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Discrete solution with o in the interior

- $\mu = \sum_{i=1}^m \lambda_i \delta_{u_i}$: $\lambda_i > 0$, $i = 1, \dots, m$, and $\{u_1, \dots, u_m\} \subset S^{n-1}$ not contained in a closed hemisphere.
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Then there exist a convex polytope $P \in \mathscr{K}^n_{(o)}$ and $\tau < 0$ such that

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$$\|h_K\|_{\mu,\varphi} := \inf \left\{ \lambda > 0 : \frac{1}{\varphi(1)\, \mu(S^{n-1})} \int_{S^{n-1}} \varphi\left(\frac{h_K(u)}{\lambda} \right) d\mu(u) \le 1 \right\}.$$

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Theorem

Under the conditions above, there exists a $K \in \mathcal{K}_o^n$ with $\mathrm{int}K \neq \emptyset$ such that

$$(\psi(h_K))\mu = \left(\int_{S^{n-1}} \psi(h_K(u))\, d\mu(u) \right) \frac{\widetilde{C}_G(K,\cdot)}{\widetilde{C}_G(K,S^{n-1})}.$$

Remarks

- Considering μ to be discrete has the following advantages: to transfer the optimization problem on functions into a multivariate optimization problem; and can obtain more information on the solutions must be polytopes, such as the origin lie in the interiors; etc.

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- $\psi = t^p, G = t^q$: $p > 1, q < 0$, our results recover the solution of Böröczky and Fodor's result.

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Theorem

Under the conditions above, there exists a $K\in\mathscr{K}^n_{(o)s}$ (symmetric convex bodies) with $\mathrm{int}K\neq\emptyset$ such that

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Thank you very much!!!