Families of well-approximable measures

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Joint with Max Goering and Christian Weiß

Overview

1. Dimension 1: Lebesgue is hardest to approximate
2. Open question for $d \geq 2$
3. Family of measures with better rates for $d \geq 2$
4. Changing the metric? Combinatorial methods?

Star Discrepancy

$$D_N^*(\mu, \nu) = \sup_{A \in \mathcal{A}} |\mu(A) - \nu(A)|$$

where $\mathcal{A}$ set of all half-open axis-parallel boxes in $[0, 1]^d$ with one vertex at the origin.

Theorem (Old news)

$\lambda_1 =$ *Lebesgue measure on* $[0,1]$

- *For all* $N \in \mathbb{N}$ *there exists a finite set* $(x_i)_{i=1}^N$ *so that*

$$D_N^*\left(\lambda_1, \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}\right) \leq \frac{c}{N}$$

  c *independent of* N.

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- *For any finite set* $(x_i)_{i=1}^N$,

$$D_N^*\left(\lambda_1; \frac{1}{N}\sum_{i=1}^N \delta_{x_i}\right) \geq \frac{1}{2N}.$$

Theorem (Renewed News: Fairchild–Goering–Weiss 2020)

μ *normalized Borel measure on* $[0, 1]$ *with Lebesgue decomposition*

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- *If $\mu_d = 0$ For any finite set $(x_i)_{i=1}^N$,*

$$D_N^*\left(\mu; \frac{1}{N}\sum_{i=1}^N \delta_{x_i}\right) \geq \frac{1}{2N}.$$

Generalized ideas of Hlawka, Mück 1972

(Aistleitner, Bilyk, Nikolov 2017)

- For $d \geq 1$
- there exists c_d so that
- for all $N \geq 2$
- for all μ Borel measure on $[0,1]^d$
- there exists points $x_1, \ldots, x_N \in [0,1]^d$ so that

$$D_N^* \left(\mu; \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \right) \leq c_d \frac{(\log N)^{d-\frac{1}{2}}}{N}$$

Open Question (Aistleitner, Bilyk, Nikolov 2017)

- For $d \geq 1$
- does there exist μ Borel measure on $[0,1]^d$
- for all $N \geq 2$
- so that there exists points $x_1, \ldots, x_N \in [0,1]^d$ so that

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When $d=1$ FGW2020 confirms no such μ exists.

Theorem (FGW202 response to open question for $d \geq 2$)

- *For* $d \geq 1$
- *there is a family of discrete uniform Borel measures* μ *on* $[0,1]^d$
- *for all* $N \geq 2$
- *so that there exists points* $x_1, \ldots, x_N \in [0,1]^d$ *so that*

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- Proof uses total variation metric.
- Sufficient assumptions on family of measures

$$\mu = \sum_{j=1}^{\infty} \alpha_j \delta_{y_j}$$

with $\alpha_j \leq r^{j-1}\alpha_1$ for $0 < r < 1$ and $c = c_r$.

Theorem (FGW2020 response to open question for $d \geq 2$)

- *For* $d \geq 1$
- *there is a family of discrete probability measures* μ *on* $[0,1]^d$
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Proof Idea

- Approximate μ by measures supported on a finite set using decay rate on the tail.
- Explicitly construct sets $x_1, \ldots, x_N$ approximating a given finitely supported measure

Further Directions

- Use combinatorial methods for larger families of discrete measures?

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- Wasserstein metric?

Theorem (Steinerberger 2018)

If α is a badly approximable number, then

$$W_2\left(\frac{1}{N}\sum_{n=1}^{N}\delta_{n\alpha}, \lambda_1\right) \leq c_\alpha \frac{(\log N)^{\frac{1}{2}}}{N}.$$