25 interesting problems on χ

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New perspectives in coloring and structure
March 15-20 2020

Notions and notations

Graphs - finite and simple
$\chi(G)$ - chromatic number of G
$\omega(G)$ - clique number of G
$h(G)$ - Hadwiger number of G
$\tau(G)$ - number of spanning trees of G
Hole - induced cycle of length at least 4
H-free - not containing H as an induced subgraph
$\mathcal{H}$-free - H-free for every $H \in \mathcal{H}$

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What is the optimal χ-bounding function for the class?

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$z(G)$ is the cochromatic number of G, the smallest k such that $V(G)$ can be partitioned into k sets, each of which is either a clique or a stable set.
This is a weakening of Hadwiger's conjecture since $z(G) \leq \chi(G)$.

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Determining the structure of such graphs for some specific small T would be interesting. For example, $T = P_7$.

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τ enjoys the Deletion-contraction property: $\tau(G) = \tau(G/e) + \tau(G\backslash e)$.
Hence the inequality is a consequence of Hadwiger's conjecture.

Problem 9

PROBLEM 9. Conjecture: If G is (P_5, C_5)-free, then $\chi \leq \omega^2$.

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Problem 10

PROBLEM 10. Conjecture: Every 7-chromatic graph contains either a path, cycle or clique on 5 vertices as an induced subgraph.

Problem 11

PROBLEM 11. What is the optimal χ-bounding function for the class of graphs whose complements have girth at least 6?

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Such graphs are $2K_2$-free, hence quadratically χ-bounded

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Similarly, χ-boundedness of the class of graphs without holes of length at least 5 is known but polynomial χ-boundedness is open.
Structure/construction/decomposition for such graphs would be very interesting.

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Kuhn-Osthus proved: If G is $(C_3, C_4, C_5, \cdots C_{27})$-free, then $h(G) \geq \chi(G)$.

Problem 14

PROBLEM 14. For every tree T, the class of (T, C_3, C_5)-free graphs has bounded χ.

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PROBLEM 15. Conjecture: $\chi \leq \omega^{l-1}$.

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Here l is the length of a longest induced path in G.

Problem 16

PROBLEM 16. If neither G nor $\overline{G}$ contains a K_t-minor, then $\chi(G) \leq t - 1$.

Problem 17

PROBLEM 17. The only 6-chromatic graph with the property that deleting the ends of any edge results in a 4-chromatic graph is K_6.

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This is the first open case of the so-called Lovasz' double-critical graph conjecture.

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Problem 19

PROBLEM 19. For every t, the class of (P_t, C_4)-free graphs is linearly χ-bounded.

Problem 20

PROBLEM 20. For every t, the class of (P_t, C_5)-free graphs is polynomially χ-bounded.

Problem 21

PROBLEM 21. Let G be a graph such that $\chi(H) \leq \omega(H) + 2$ for every induced subgraph H of G. Then $h(G) \geq \chi(G)$.

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More generally: Fix a positive integer k. Let G be a graph such that $\chi(H) \leq \omega(H) + k$ for every induced subgraph H of G. Then $h(G) \geq \chi(G)$.
Proving for all k is proving Hadwiger's conjecture. What is the largest value of k for which this approach looks promising?

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Problem 24

PROBLEM 24. For every t, the class of $(P_t, \overline{P_5})$-free graphs is polynomially χ-bounded.

Problem 25

PROBLEM 25. The class of graphs not containing an induced cycle of length a power of a prime has bounded χ.

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Hopefully this should be easier than just excluding prime lengths which is also open.

THANKS FOR YOUR ATTENTION.