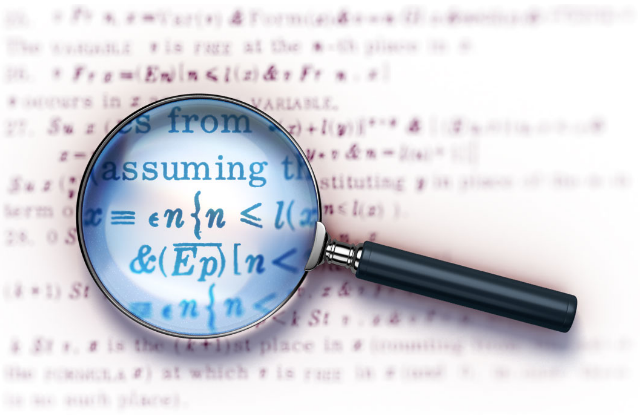

Automated Proof Search: The Aftermath

Susanna de Rezende, Mika Göös, Sajin Koroth, Ian Mertz, Jakob Nordström, Toni Pitassi, Robert Robere, Dmitry Sokolov

Me me me!

Me me me!

Here is a problem in
Proof Complexity

Me me me!

Here is a problem in
Proof Complexity

But I wanna work on
Communication :(

Me me me!

Lifting for dag-like models?

Me me me!

Lifting for dag-like models?

Great idea!

Me me me!

Lifting for dag-like models?

Great idea!

Takeaway: **Monotone circuits** for XOR-SAT can simulate **Resolution**

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Takeaway: **Monotone circuits** for XOR-SAT can simulate **Resolution**

. . . we proved the converse [GGKS'18]

Me me me!

Lifting for dag-like models?

Great idea!

Takeaway: **Monotone circuits** for Xor-Sat can simulate **Resolution**

…we proved the converse [GGKS'18]

$\Longrightarrow$ Proof complexity is **cool**!

This talk: Results on

Hardness of Automatability

- **Simpler proof** for Resolution [Atserias–Müller'19]
- **Generalises better:** **NP**-hardness for
  - Nullstellensatz . . . *previously* [Galesi–Lauria'10]
  - Polynomial Calculus . . . *previously* [Galesi–Lauria'10]
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  - Cutting Planes (requires more work)

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  - ***Still open:*** *Sum-of-Squares*

Simple proof of
[Atserias–Müller'19]

Atserias–Müller

There is polytime reduction $\mathcal{A}$ that maps n-variate CNF F to unsatisfiable CNF $\mathcal{A}(F)$:

$$F \text{ is } \textbf{SAT} \implies \mathcal{A}(F) \text{ has Resolution length } n^{O(1)}$$
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Overview of $\mathcal{A}$:

Input: F that is SAT-vs-UNSAT

1. Construct $\mathsf{Ref}(F)$ of block-width $O(1)$-vs-$n^{\Omega(1)}$
2. Output $\mathsf{Lifted\text{-}Ref}(F)$ of Res-length $n^{O(1)}$-vs-$2^{n^{\Omega(1)}}$

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Overview of $\mathcal{A}$:

We simplify

Input: F that is SAT-vs-UNSAT

1. Construct Ref(F) of block-width $O(1)$-vs-$n^{\Omega(1)}$
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Key: Reduction from PHP

(When F is UNSAT)

$$\underset{\text{width}}{\text{PHP}} \leq \underset{\text{block-width}}{\text{Ref}(F)}$$

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Reduction via **Tree-Resolution**
…in depth n^{ε} (surprising!)

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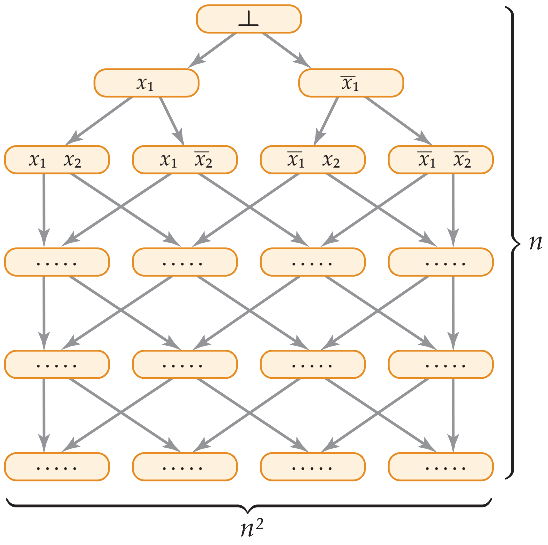
Reduction via **Tree-Resolution**
. . . in depth n^{ε} (surprising!)

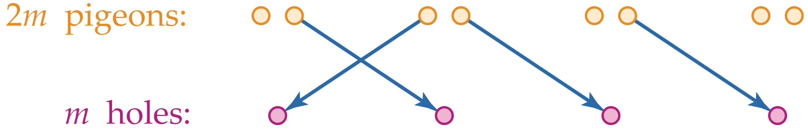
$$\text{width}(\mathrm{PHP})/n^{\varepsilon} \;\le\; \text{block-width}(\mathrm{Ref}(F))$$

Ref(F)

- Encoding of "F admits short Resolution proof"
- Consists of **blocks**
  n layers of n^2 blocks
- **Blocks** encode clauses
  - Indicators for literals
  - Pointers to children
  - Name of axiom of F
- **Important:** Children picked from lower layer
  $\Longrightarrow$ **Dag!**



PHP: Weak bit-encoded invertible function



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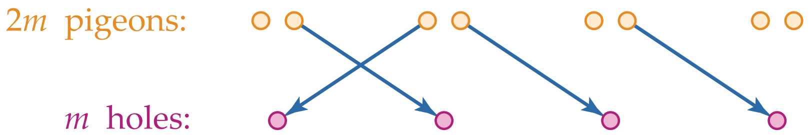

Bit + inv: Each pigeon (hole) associated with $O(\log m)$ variables that *name* one hole (pigeon)

$i \to j$ *iff* i names j and vice versa

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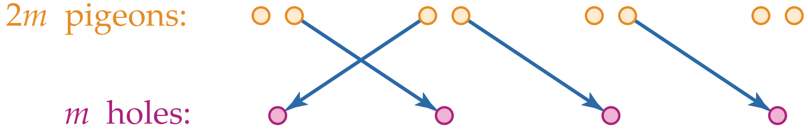
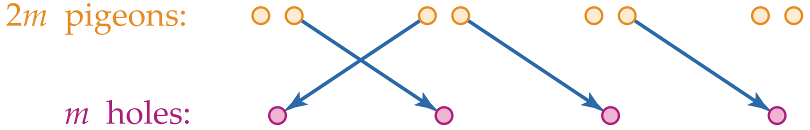


Bit + inv: Each pigeon (hole) associated with $O(\log m)$ variables that *name* one hole (pigeon)

$i \rightarrow j$ *iff* i names j and vice versa

Function: Require every pigeon maps to hole (mapping need not be *onto*)

PHP: Weak bit-encoded invertible function



Lower bounds

PHP_m^{2m} requires degree $\Omega(m)$ for

- Polynomial Calculus [Razborov'98]
- Sherali–Adams [Georgiou–Magen'08]

(Using unary encodings. Easy for SoS)

$$\underset{\text{width}}{\mathrm{PHP}} \leq \underset{\text{block-width}}{\mathrm{Ref}(F)}$$

Intution: $\mathrm{Ref}(F)$ looks locally like **full binary tree**

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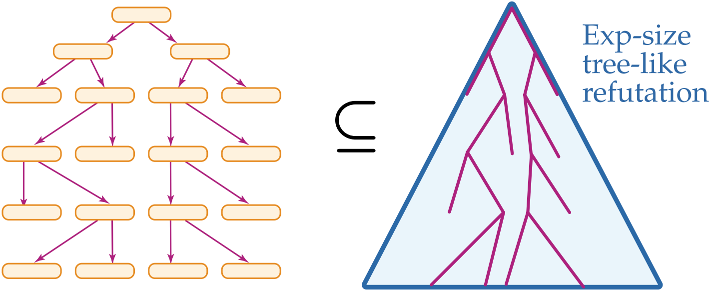

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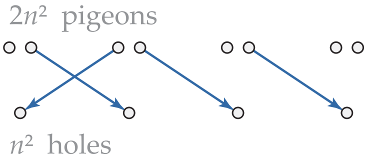

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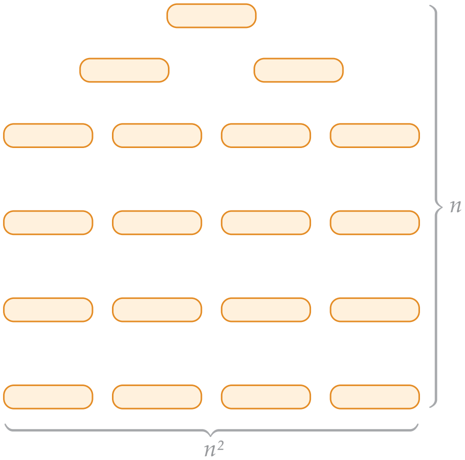

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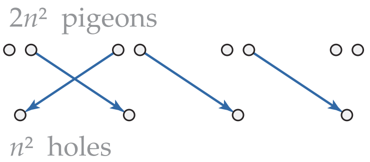

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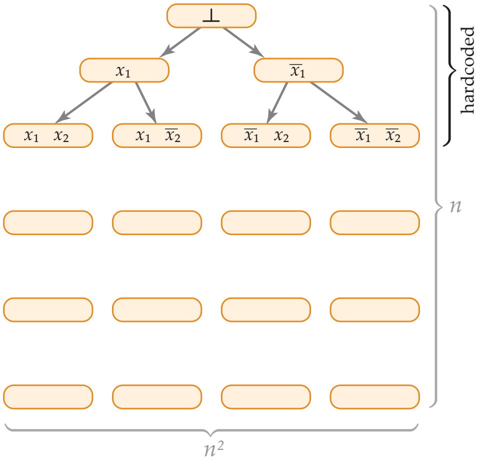

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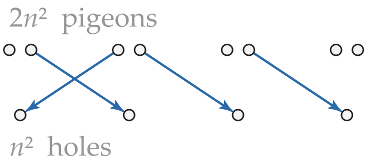

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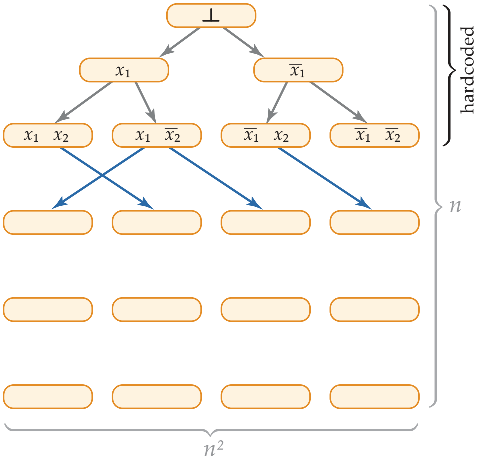

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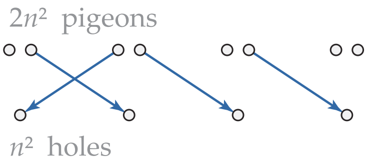

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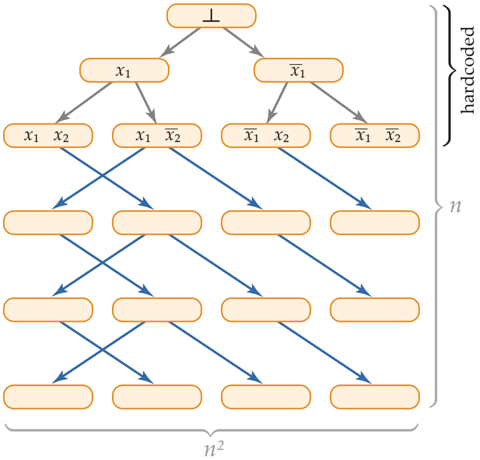

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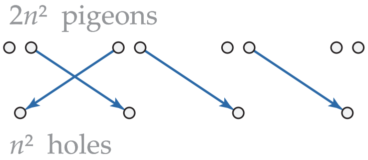

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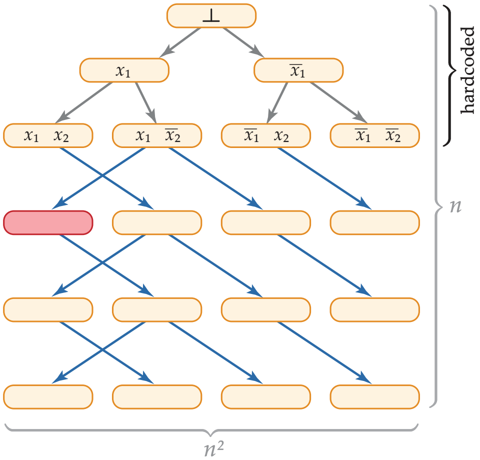

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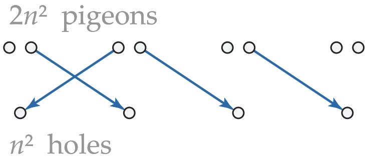

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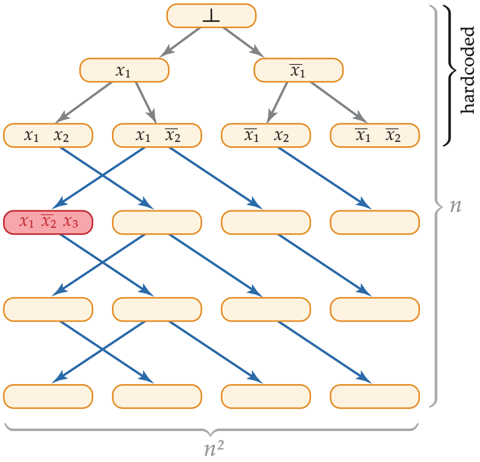

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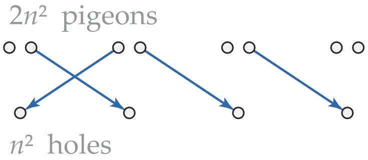

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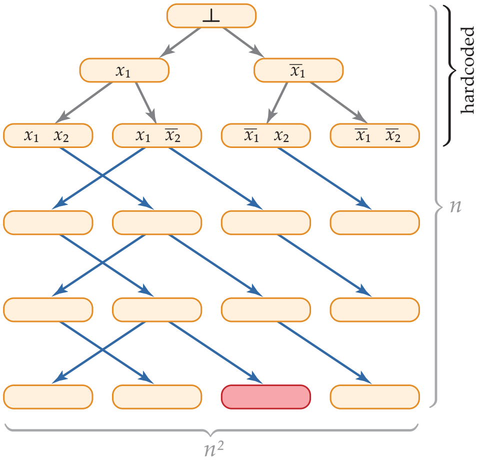

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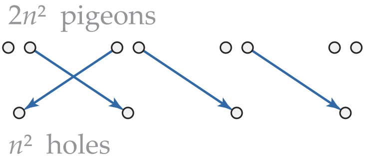

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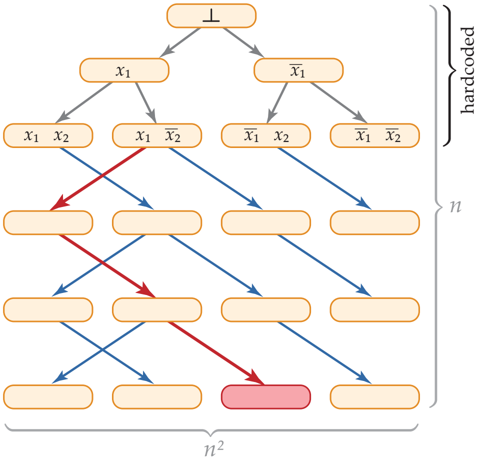

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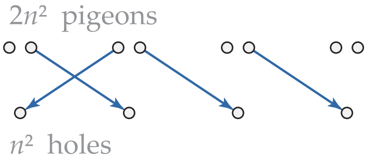

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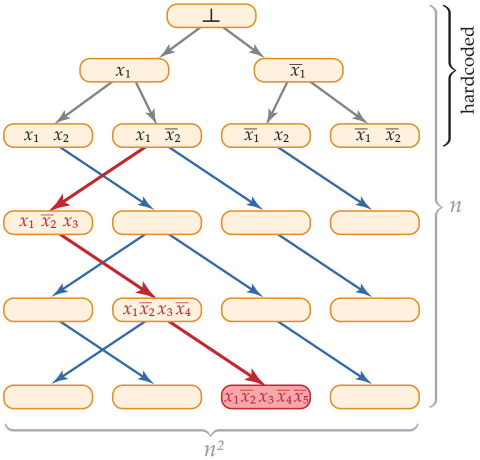

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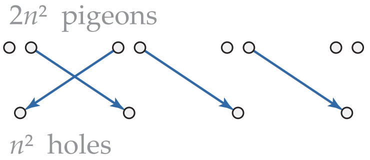

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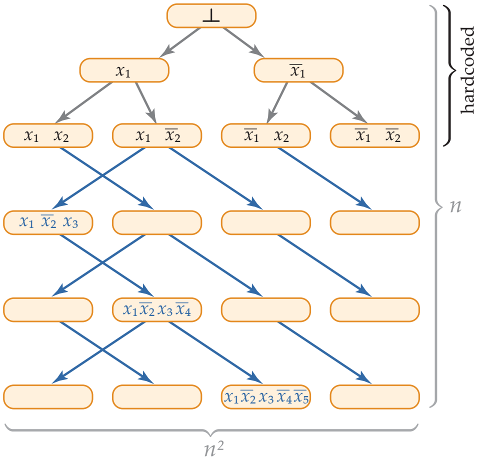

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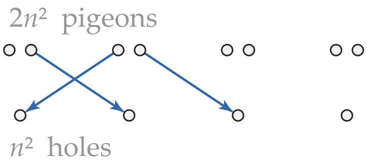

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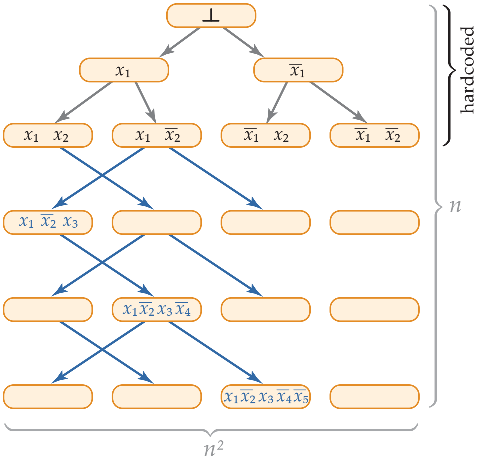

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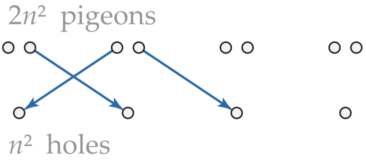

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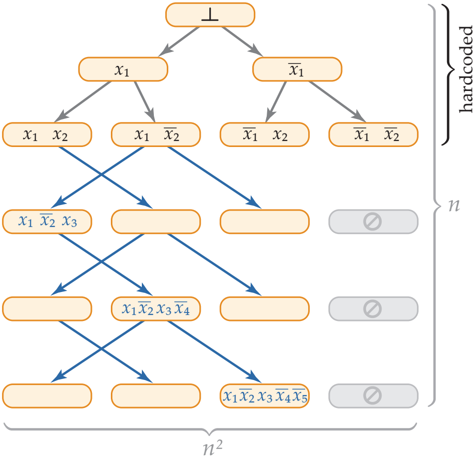

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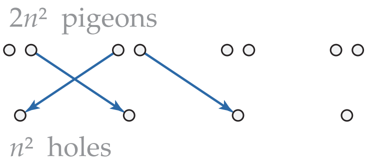

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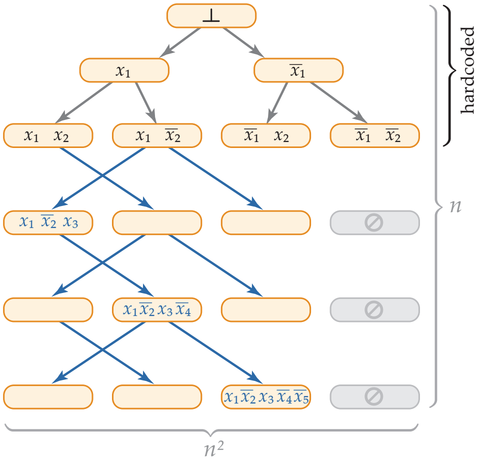

Rules of the game

- Each var of $\mathrm{Ref}(F)$ is decision tree of vars of PHP
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Conclusion: $\text{block-width}(\mathrm{Ref}(F)) \cdot n \;\geq \text{width}(\textbf{PHP}) = \Omega(n^2)$

$\Longrightarrow \quad \text{block-width}(\mathrm{Ref}(F)) \geq \Omega(n)$

When F is UNSAT

$$\underset{\text{width}}{\mathrm{PHP}} \quad \leq \quad \underset{\text{block-width}}{\mathrm{Ref}(F)}$$

We showed: $\mathrm{Ref}(F)$ has block-width $n^{\Omega(1)}$

Apply lifting: Lifted-$\mathrm{Ref}(F)$ has Resolution size $2^{n^{\Omega(1)}}$

Atserias–Müller

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Our extension

There is polytime reduction $\mathcal{A}$:

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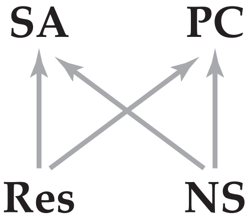

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Upper bound for NS

$\mathrm{Ref}(F)$

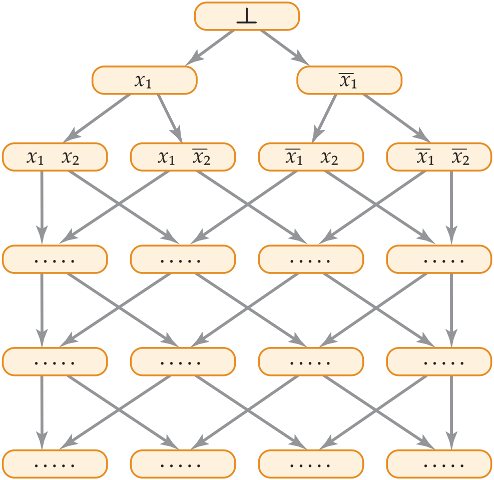
F satisfied by $x \in \{0,1\}^n$

Easy for Resolution since

$\mathrm{Ref}(F) \leq$ Pebbling

Pebbling

- Root of DAG pebbled
- If node is pebbled, then ≥ 1 children is pebbled



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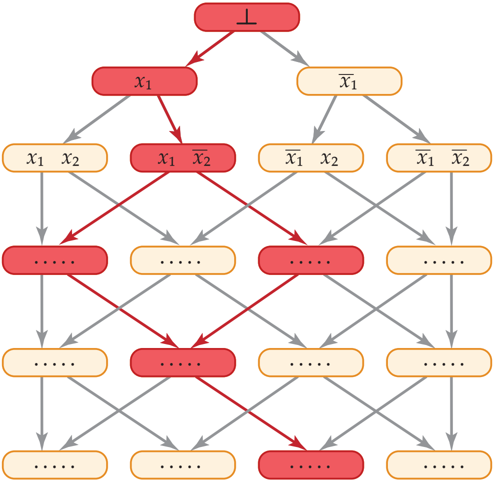
“Pebbled” block

$=$ falsified by x



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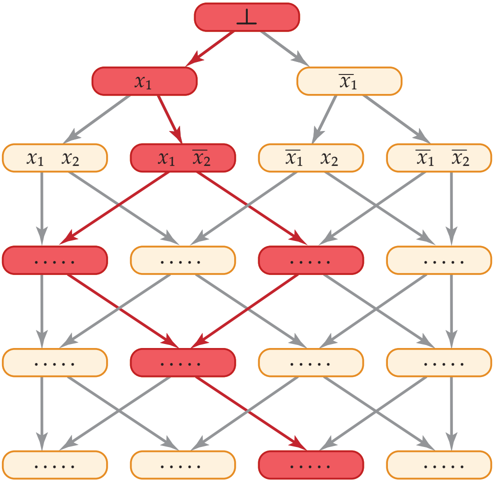
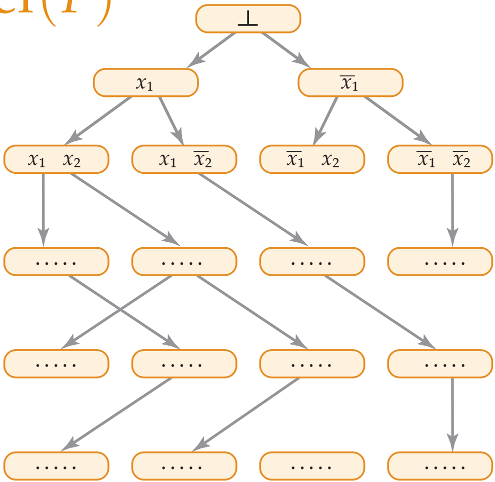
…but Pebbling is
hard for **NS**!



Solution: TreeRef(F)

F satisfied by $x \in \{0,1\}^n$



Solution: TreeRef(F)

F satisfied by $x \in \{0,1\}^n$

Easy for NS since

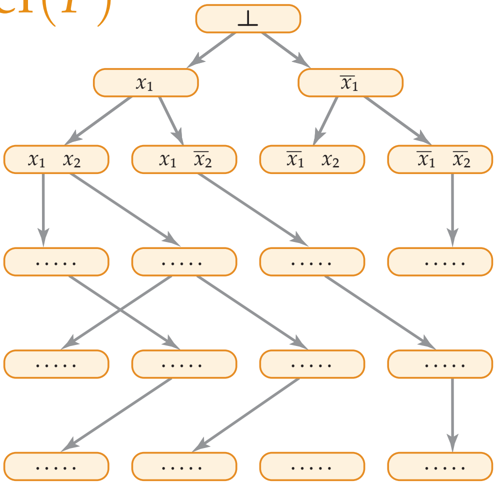
TreeRef(F)

$\leq$ End-of-Line

aka Onto-PHP

End-of-Line

- Root pebbled
- If node is pebbled, then unique child and unique parent pebbled



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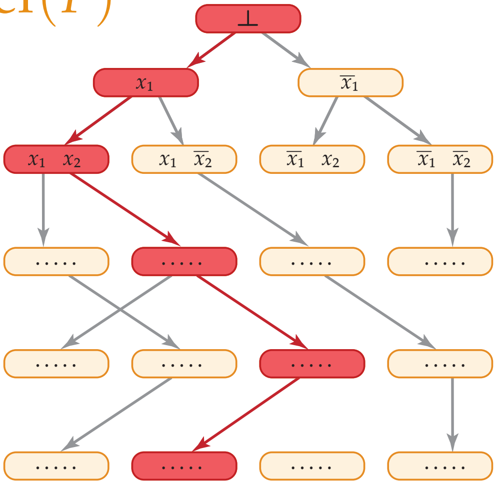

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Solution: TreeRef(F)

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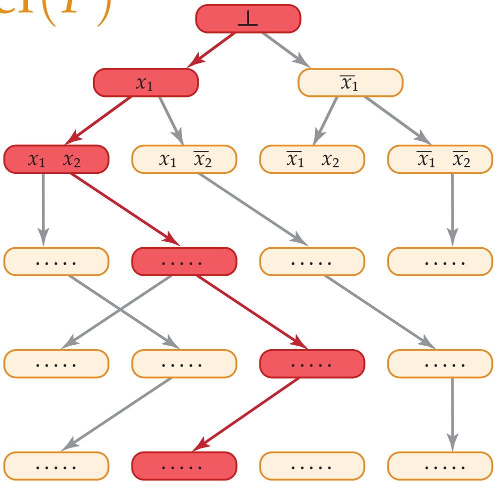
Hardness still works!



Cutting Planes

Result for Cutting Planes

There is polytime reduction $\mathcal{A}$:

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$$F \text{ is } \mathbf{UNSAT} \implies \mathcal{A}(F) \text{ has } \mathbf{CP} \text{ length } 2^{n^{\Omega(1)}}$$

Highlights

- [GGKS'18]: F has width $w \Rightarrow F \circ g$ has CP length $2^{\Omega(w)}$
- Instead: need **block**-lifting
- Bypass monotone circuits (first such technique?)

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Atserias, Müller

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Cheers!


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