Dynamics of plane partitions

Oliver Pechenik
University of Waterloo

21 Oct 2020

Based on joint work with Becky Patrias (St. Thomas)
arXiv:2003.13152
This talk is being recorded

Rowmotion

- Let P be a finite poset and $J(P)$ its set of order ideals.
- **Rowmotion** is the permutation $\psi \colon J(P) \to J(P)$ sending an order ideal I to the smallest order ideal $\psi(I)$ containing the minimal elements of $P \setminus I$.

Rowmotion

- Let P be a finite poset and $J(P)$ its set of order ideals.
- **Rowmotion** is the permutation $\psi : J(P) \to J(P)$ sending an order ideal I to the smallest order ideal $\psi(I)$ containing the minimal elements of $P \setminus I$.
- **Q:** What is the orbit structure of $J(P)$ under iterating ψ?
- For general P, it's a mess!
- But for your favorite P, there is probably a lot of structure

Rowmotion

- Let P be a finite poset and $J(P)$ its set of order ideals.
- **Rowmotion** is the permutation $\psi\colon J(P) \to J(P)$ sending an order ideal I to the smallest order ideal $\psi(I)$ containing the minimal elements of $P \setminus I$.
- **Q:** What is the orbit structure of $J(P)$ under iterating ψ?
- For general P, it's a mess!
- But for your favorite P, there is probably a lot of structure
- This morning: $P = \mathbf{a} \times \mathbf{b} \times \mathbf{c}$, a product of three chain posets.

$c = 1$

Theorem (Brouwer+Schrijver 1974)

The order of ψ on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$ is $a + b$.
More precisely, for $I \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{1})$, $|\mathcal{O}(I)|$ divides $a + b$ and $|\mathcal{O}(\emptyset)| = a + b$.

Theorem (Striker+Williams 2012)

$\big(J(\mathbf{a} \times \mathbf{b} \times \mathbf{1}), \psi, f(q)\big)$ exhibits cyclic sieving, where $f(q)$ is the q-enumerator for order ideals by cardinality.

Theorem (Propp+Roby 2015, Buch+Wang 2019)

For each orbit $\mathcal{O}(I)$, the average order ideal size is $\frac{ab}{2}$.

$c = 2$

Theorem (Cameron+Fon Der Flaass 1995)

The order of ψ *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$ *is* $a + b + 1$.
More precisely, for $I \in J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$, $|\mathcal{O}(I)|$ *divides* $a + b + 1$ *and* $|\mathcal{O}(\emptyset)| = a + b + 1$.

Theorem (Striker+Williams 2012, Rush+Shi 2013)

$\big(J(\mathbf{a} \times \mathbf{b} \times \mathbf{2}), \psi, f(q)\big)$ *exhibits cyclic sieving, where* $f(q)$ *is the q-enumerator for order ideals by cardinality.*

Theorem (Vorland 2019)

For each orbit $\mathcal{O}(I)$, *the average order ideal size is* $\frac{ab}{1}$.

Conjecture (Dilks+P+Striker 2017)

The order of ψ on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{3})$ is $a + b + 2$.

No obvious CSP

No obvious homomesy

$c \geq 4$

Order of ψ generally greater than $a + b + c - 1$ but unknown.

No good bounds on order known. (For $a = b = c = 4$, order is $11 \cdot 3$; for $a = 4$, $b = c = 11$, order is $\geq 309 \cdot 25$.)

No obvious CSP

No obvious homomesy

Cameron+Fon-Der-Flaass Conjecture

Conjecture (Cameron+Fon-der-Flaass 1995)

If $a + b + c - 1$ is prime, then $a + b + c - 1$ divides every ψ-orbit size on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$.

Cameron+Fon-Der-Flaass Conjecture

Conjecture (Cameron+Fon-der-Flaass 1995)

If $a+b+c-1$ is prime, then $a+b+c-1$ divides every ψ-orbit size on $J(\mathbf{a}\times\mathbf{b}\times\mathbf{c})$.

Theorem (Cameron+Fon-der-Flaass 1995)

The conjecture holds when $c > ab-a-b+1$.

Cameron+Fon-Der-Flaass Conjecture

Conjecture (Cameron+Fon-der-Flaass 1995)

If $a+b+c-1$ is prime, then $a+b+c-1$ divides every ψ-orbit size on $J(\mathbf{a}\times\mathbf{b}\times\mathbf{c})$.

Theorem (Cameron+Fon-der-Flaass 1995)

The conjecture holds when $c > ab-a-b+1$.

Theorem (Dilks+P+Striker 2017)

The conjecture holds when $c > \frac{2ab-2}{3} - a - b + 2$.

Cameron+Fon-Der-Flaass Conjecture

Conjecture (Cameron+Fon-der-Flaass 1995)

If $a + b + c - 1$ *is prime, then* $a + b + c - 1$ *divides every* ψ*-orbit size on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$.

Theorem (Cameron+Fon-der-Flaass 1995)

The conjecture holds when $c > ab - a - b + 1$.

Theorem (Dilks+P+Striker 2017)

The conjecture holds when $c > \frac{2ab-2}{3} - a - b + 2$.

Theorem (Patrias+P 2020)

The conjecture is true. More generally, with no primality condition, we have

$$\gcd(a + b + c - 1, |\mathcal{O}(I)|) > 1$$

K-jeu de taquin

Let $\lambda \subseteq \nu$ be partitions. An **increasing tableau** of shape $\nu \setminus \lambda$ is a filling of the skew Young diagram $\nu \setminus \lambda$ by positive integers with strictly increasing rows and columns.

$\lambda = (3,2,1), \nu = (4,4,3)$, $T =$

			1
		2	3
	2	4	

K**-theoretic jeu de taquin** (Thomas+Yong 2009) rectifies this to an increasing tableau of partition shape (computing K-theoretic Schubert structure coefficients).

K-rectification of skew increasing tableaux

			1
		2	3
	2	4	

K-rectification recipe (Thomas+Yong 2009, Buch+Samuel 2016):

K-rectification of skew increasing tableaux

		•	1
	•	2	3
•	2	4	

K-rectification recipe (Thomas+Yong 2009, Buch+Samuel 2016):

1. Fill all inner corners with •.

K-rectification of skew increasing tableaux

		$\bullet$	1
	$\bullet$	2	3
$\bullet$	2	4	

K-rectification recipe (Thomas+Yong 2009, Buch+Samuel 2016):

1. Fill all inner corners with $\bullet$.
2. Apply $\updownarrow^{1}_{\bullet}$

The swap operator $\updownarrow^{\heartsuit}_{\spadesuit}$ turns each $\spadesuit$ next to a $\heartsuit$ into $\heartsuit$, and each $\heartsuit$ next to $\spadesuit$ into $\spadesuit$.

K-rectification of skew increasing tableaux

		1	$\bullet$
	$\bullet$	2	3
$\bullet$	2	4	

K-rectification recipe (Thomas+Yong 2009, Buch+Samuel 2016):

1. Fill all inner corners with $\bullet$.
2. Apply $\updownarrow^{1}_{\bullet}$
3. Apply $\updownarrow^{2}_{\bullet}$, $\updownarrow^{3}_{\bullet}$, ...

The swap operator $\updownarrow^{\heartsuit}_{\spadesuit}$ turns each $\spadesuit$ next to a $\heartsuit$ into $\heartsuit$, and each $\heartsuit$ next to $\spadesuit$ into $\spadesuit$.

K-rectification of skew increasing tableaux

		1	$\bullet$
	2	$\bullet$	3
2	$\bullet$	4	

K-rectification recipe (Thomas+Yong 2009, Buch+Samuel 2016):

1. Fill all inner corners with $\bullet$.
2. Apply $\updownarrow^{1}_{\bullet}$
3. Apply $\updownarrow^{2}_{\bullet}$, $\updownarrow^{3}_{\bullet}$, . . .

The swap operator $\updownarrow^{\heartsuit}_{\spadesuit}$ turns each $\spadesuit$ next to a $\heartsuit$ into $\heartsuit$, and each $\heartsuit$ next to $\spadesuit$ into $\spadesuit$.

K-rectification of skew increasing tableaux

		1	3
	2	3	$\bullet$
2	$\bullet$	4	

K-rectification recipe (Thomas+Yong 2009, Buch+Samuel 2016):

1. Fill all inner corners with $\bullet$.
2. Apply $\updownarrow_{\bullet}^{1}$
3. Apply $\updownarrow_{\bullet}^{2}$, $\updownarrow_{\bullet}^{3}$, . . .

The swap operator $\updownarrow_{\spadesuit}^{\heartsuit}$ turns each $\spadesuit$ next to a $\heartsuit$ into $\heartsuit$, and each $\heartsuit$ next to $\spadesuit$ into $\spadesuit$.

K-rectification of skew increasing tableaux

		1	3
	2	3	$\bullet$
2	4	$\bullet$	

K-rectification recipe (Thomas+Yong 2009, Buch+Samuel 2016):

1. Fill all inner corners with $\bullet$.
2. Apply $\updownarrow_{\bullet}^{1}$
3. Apply $\updownarrow_{\bullet}^{2}$, $\updownarrow_{\bullet}^{3}$, . . .
4. Delete outer $\bullet$s.

The swap operator $\updownarrow_{\spadesuit}^{\heartsuit}$ turns each $\spadesuit$ next to a $\heartsuit$ into $\heartsuit$, and each $\heartsuit$ next to $\spadesuit$ into $\spadesuit$.

K-rectification of skew increasing tableaux

		1	3
	2	3	
2	4		

K-rectification recipe (Thomas+Yong 2009, Buch+Samuel 2016):

1. Fill all inner corners with $\bullet$.
2. Apply $\updownarrow^1_\bullet$
3. Apply $\updownarrow^2_\bullet$, $\updownarrow^3_\bullet$, ...
4. Delete outer $\bullet$s.
5. Repeat with new inner corners!

The swap operator $\updownarrow^\heartsuit_\spadesuit$ turns each $\spadesuit$ next to a $\heartsuit$ into $\heartsuit$, and each $\heartsuit$ next to $\spadesuit$ into $\spadesuit$.

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$$\mathrm{Inc}^5(2 \times 3) \ni T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 & 5 \\ \hline \end{array}$$

K-promotion recipe (P 2014):

1. Delete 1

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$\mathrm{Inc}^5(2 \times 3) \ni T$

	2	4
3	4	5

K-promotion recipe (P 2014):

1. Delete 1
2. Decrement each letter

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$\mathrm{Inc}^5(2 \times 3) \ni T$

	1	3
2	3	4

K-promotion recipe (P 2014):

1. Delete 1
2. Decrement each letter
3. K-rectify

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$\mathrm{Inc}^5(2 \times 3) \ni T$

•	1	3
2	3	4

K-promotion recipe (P 2014):

1. Delete 1
2. Decrement each letter
3. K-rectify

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$\mathrm{Inc}^5(2 \times 3) \ni T$

1	•	3
2	3	4

K-promotion recipe (P 2014):

1. Delete 1
2. Decrement each letter
3. K-rectify

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$\mathrm{Inc}^5(2 \times 3) \ni T$

1	3	•
2	•	4

K-promotion recipe (P 2014):

1. Delete 1
2. Decrement each letter
3. K-rectify

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$\mathrm{Inc}^5(2 \times 3) \ni T$

1	3	4
2	4	$\bullet$

K-promotion recipe (P 2014):

1. Delete 1
2. Decrement each letter
3. K-rectify

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$\mathrm{Inc}^5(2 \times 3) \ni T$

1	3	4
2	4	

K-promotion recipe (P 2014):

1. Delete 1
2. Decrement each letter
3. K-rectify
4. Fill empty cells with q

K-promotion of increasing tableaux

$\mathrm{Inc}^q(\lambda) = \{$increasing tableaux of shape λ with entries in [q]$\}$

$$\mathrm{Inc}^5(2 \times 3) \ni T \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 4 & 5 \\ \hline \end{array} = \psi(T)$$

K-promotion recipe (P 2014):

1. Delete 1
2. Decrement each letter
3. K-rectify
4. Fill empty cells with q

Equivariant bijection

Theorem (Dilks+P+Striker 2017)

There is an equivariant bijection between

$$\big(J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}), \psi\big)$$

and

$$\big(\mathrm{Inc}^{a+b+c-1}(a \times b), \psi\big)$$

Why does this help?

Why does this help?

1	2	5
3	5	7

- Easier to compute by hand
- Tools from K-theoretic Schubert calculus
- Can focus on **gapless** tableaux (all elements of $[q]$ appear)
  - If $T \in \mathrm{Inc}^q(\lambda)$ has q' distinct labels, its **deflation** $T' \in \mathrm{Inc}^{q'}(\lambda)$ is obtained by replacing the ith smallest entries of T with i.
  - The **content** of T is the binary string of length q recording which elements of $[q]$ appear in T.

Gapless tableaux

Theorem (Mandel+P 2018)

Let $T \in \mathrm{Inc}^q(\lambda)$ and let $T' \in \mathrm{Inc}^{q'}(\lambda)$ be its deflation. Suppose ψ has order τ' on T' and cyclic shift has order ℓ on the content of T. Then on T, ψ has order

$$\tau = \frac{\ell \tau'}{\gcd(\ell q'/q, \tau')}$$

Example:

$$J(\mathbf{8} \times \mathbf{8} \times \mathbf{c}) \xrightarrow{\sim} \mathrm{Inc}^{15+c}(8 \times 8) \twoheadrightarrow \mathrm{Inc}_{\mathrm{gl}}(8 \times 8)$$

$c = 1$

$$J(\mathbf{a} \times \mathbf{b} \times \mathbf{1}) \xrightarrow{\sim} \mathrm{Inc}^{a+b}(1 \times a) \twoheadrightarrow \mathrm{Inc}_{gl}(1 \times a)$$

1	2	3	4	5	6	7

For each a, there is a unique such tableau. It is fixed by ψ.

Theorem

The order of ψ on $\mathrm{Inc}^q(1 \times a)$ is q.

$$J(\mathbf{a} \times \mathbf{b} \times \mathbf{1}) \xrightarrow{\sim} \mathrm{Inc}^{a+b}(a \times b)$$

Theorem (Dilks+P+Striker 2017)

The order of ψ on $\mathrm{Inc}^{a+b}(a \times b)$ is $a + b$.

$c = 2$

$$\mathrm{Inc}^{a+b+1}(a \times b) \xleftarrow{\sim} J(\mathbf{a} \times \mathbf{b} \times \mathbf{2}) \xrightarrow{\sim} \mathrm{Inc}^{a+b+1}(2 \times a)$$

Theorem (P 2014, Dilks+P+Striker 2017)

The order of ψ on $\mathrm{Inc}^q(2 \times a)$ is q.

Theorem (Dilks+P+Striker 2017)

The order of ψ on $\mathrm{Inc}^{a+b+1}(a \times b)$ is $a + b + 1$.

Theorem (P 2014, Rhoades 2017)

$\left(\mathrm{Inc}^q_{\mathrm{gl}}(2 \times a), \psi, f(t)\right)$ exhibits cyclic sieving, where $f(t)$ is the t-enumerator for $\mathrm{Inc}^q_{\mathrm{gl}}(2 \times a)$ by major index.

Theorem (Bloom+P+Saracino 2016)

Fix $S \subseteq 2 \times a$, fixed by 180° rotation. For each orbit $\mathcal{O}(T)$ in $\mathrm{Inc}^q(2 \times a)$, the average sum of the entries of S is $(q+1)\frac{|S|}{2}$.

Frames of increasing tableaux

The **frame** of $T \in \mathrm{Inc}^q(a \times b)$ is the union of the boxes in the first/last row and the first/last column.

Example

If $T =$

1	2	4	7
3	5	6	8
5	7	8	10
7	9	10	11

, then $\psi^{11}(T) =$

1	2	4	7
3	4	6	8
5	6	8	10
7	9	10	11

.

The least k such that $\psi^k(T) = T$ is $k = 33$.

Theorem (P 2017)

For $T \in \mathrm{Inc}^q(m \times n)$, we have $\mathrm{Frame}(\psi^q(T)) = \mathrm{Frame}(T)$.

Theorem (P 2017)

Fix $S \subseteq \mathrm{Frame}(a \times b)$, fixed by 180° rotation. For each orbit $\mathcal{O}(T)$ in $\mathrm{Inc}^q(a \times b)$, the average sum of the entries of S is $(q+1)\frac{|S|}{2}$.

The Cameron+Fon-Der-Flaass Conjecture

Theorem (Patrias+P 2020)

Suppose the ψ-orbit of $T \in \mathrm{Inc}^q(a \times b)$ has cardinality k. Then k shares a prime divisor with q. (Unless $q = a + b - 1$, in which case $k = 1$.)

The Cameron+Fon-Der-Flaass Conjecture

Theorem (Patrias+P 2020)

Suppose the ψ-orbit of $T \in \mathrm{Inc}^q(a \times b)$ has cardinality k. Then k shares a prime divisor with q. (Unless $q = a + b - 1$, in which case $k = 1$.)

Proof.

Suppose $\gcd(k, q) = 1$. Then by the frame theorem, $\mathrm{Frame}(T) = \mathrm{Frame}(\psi(T))$. By analysis of the promotion operator, every frame box of such a tableau must participate in a swap, so the frame entries increase consecutively from upper-left to lower-right. So T is the unique element of $\mathrm{Inc}^{a+b-1}(a \times b)$. □

The Cameron+Fon-Der-Flaass Conjecture

Theorem (Patrias+P 2020)

Suppose the ψ-orbit of $T \in \mathrm{Inc}^q(a \times b)$ has cardinality k. Then k shares a prime divisor with q. (Unless $q = a + b - 1$, in which case $k = 1$.)

Proof.

Suppose $\gcd(k, q) = 1$. Then by the frame theorem, $\mathrm{Frame}(T) = \mathrm{Frame}(\psi(T))$. By analysis of the promotion operator, every frame box of such a tableau must participate in a swap, so the frame entries increase consecutively from upper-left to lower-right. So T is the unique element of $\mathrm{Inc}^{a+b-1}(a \times b)$. □

Corollary (Patrias+P 2020, Conj: Cameron+Fon-der-Flaass 1995)

If $p = a + b + c - 1$ is prime, then the length of every ψ-orbit on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ is a multiple of p.

Open questions

Question

What multiples occur?

Open questions

Question

What multiples occur?

Question

Can we bound the orbit sizes?

Open questions

Question

What multiples occur?

Question

Can we bound the orbit sizes?

Question

Why are some multiples more common than others? There seems to be an odd preference for odd multiples. Is this a real phenomenon?

Open questions

Question

What multiples occur?

Question

Can we bound the orbit sizes?

Question

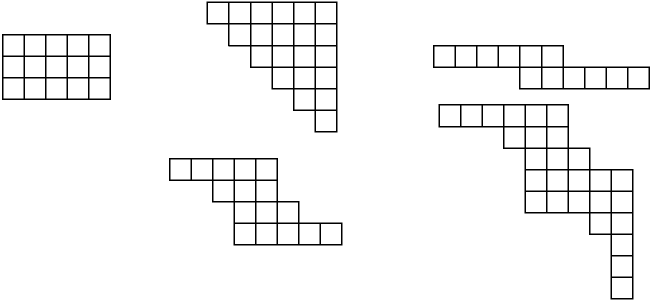
Why are some multiples more common than others? There seems to be an odd preference for odd multiples. Is this a real phenomenon?

Question

For standard tableaux, promotion orbits carry information about the geometry of Grassmannians. What is the more general geometry for plane partitions?

Other posets

- One can also consider plane partitions over bases other than rectangles.
- Especially interesting are the **minuscule** cases:



Theorem (P 2020+)

The analogue of the Cameron+Fon-Der-Flaass Conjecture holds for $M \times \mathbf{c}$, M minuscule.

Thank you!!