The ~~sandpile~~ critical groups of outerplanar graphs

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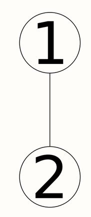
Introduction

1. Let $L(G)$ denote the Laplacian matrix of a graph G.
2. By considering $L(G)$ as a linear map $L(G): \mathbb{Z}^V \to \mathbb{Z}^V$, the cokernel of $L(G)$ is the quotient module $\mathbb{Z}^V/\mathrm{Im}\, L(G)$.
3. The torsion part of this module is the critical group $K(G)$ of G.
4. We compute the critical groups of outerplanar graphs.

Smith Normal Form

5. Two matrices M and N are said to be equivalent if there exist $P, Q \in GL_n(\mathbb{Z})$ such that $N = PMQ$.
6. Given a square integer matrix M, the Smith Normal Form (SNF) of M is the unique equivalent diagonal matrix $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ whose non-zero entries are non-negative and satisfy d_i divides d_{i+1}.
7. The diagonal elements of the SNF are known as invariant factors.
8. $\mathrm{coker}(M) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r} \oplus \mathbb{Z}^{n-r}$, where r is the rank of $n \times n$ integer matrix M.
9. Let $\Delta_k(M)$ be the gcd of the k-minors of matrix M, with $\Delta_0(M) = 1$.
10. $$d_k = \frac{\Delta_k(M)}{\Delta_{k-1}(M)}.$$
11. The reduced Laplacian matrix $L_k(G)$ is the matrix obtained by deleting the k-th row and k-th column from $L(G)$.
12. When G is connected, $K(G) \cong \mathrm{coker}(L_k(G))$

Critical ideals

13. Let $A(G)$ denote the adjacency matrix of a graph G.
14. Let $A_X(G) = \mathrm{diag}(x_1, \ldots, x_n) - A(G)$, where the variables of $X = (x_1, \ldots, x_n)$ are associated with the vertices $v_1, \ldots, v_n$ of G.
15. For $k \in [n]$, the k-th critical ideal $I_k(G)$ of G is the ideal in $\mathbb{Z}[X]$ generated by the k-minors of the matrix $A_X(G)$.
16. Let $\deg(G) = (\deg_G(v_1), \ldots, \deg_G(v_n))$.
17. Note the evaluation of $I_k(G)$ at $X = \deg(G)$ will be an ideal in $\mathbb{Z}$ generated by $\Delta_k(L(G))$.
18. 

$$A_X(P_2) = \begin{bmatrix} x_1 & -1 \\ -1 & x_2 \end{bmatrix} \qquad I_1(P_2) = \langle 1 \rangle \qquad I_2(P_2) = \langle x_1x_2 - 1 \rangle$$

Figure 1: Path with 2 vertices

19. $K(G)$ can be also obtained from the critical ideals of $G \setminus v$.

Planar graphs

20. In the following G is a plane graph, i.e., a graph already embedded on the plane.
21. Let G^* denote the dual graph of G.
22. Let G_* denote weak dual graph, defined as G^* without placing the vertex associated with the outer face.
23. $K(G) \cong K(G^*)$
24. $K(G) \cong \mathrm{coker}(L_k(G^*))$
25. Let $F_1, \ldots, F_s$ be the interior faces of G, and let $c(F_i)$ be the number of edges in the cycle bounding F_i.
26. The cycle-intersection matrix $C(G) = (c_{ij})$ is the $s \times s$ matrix such that $c_{ii} = c(F_i)$, and c_{ij} is the negative of the number of common edges in the cycles bounding F_i and F_j, for $i \neq j$.
27. $C(G)$ is a reduced Laplacian of $L(G^*)$.
28. $K(G) \cong \mathrm{coker}(C(G))$.
29. $C(G) = \mathrm{diag}(c(F_1), \ldots, c(F_s)) - A(G_*)$
30. $K(G)$ can be obtained from the critical ideals of G_*.

Outerplanar graphs

31. [1] A graph G is outerplanar if and only if it has a weak dual G_* which is a forest.
32. A graph G is biconnected outerplane if and only if its weak dual G_* is a tree.
33. Critical ideals of tree T were computed in [2] in terms of the 2-matchings of T^l, defined as T with a loop added in each vertex.
34. A 2-matching is a set of edges $\mathcal{M} \subseteq E(G)$ such that every vertex of G is incident to at most two edges in $\mathcal{M}$
35. Given a 2-matching $\mathcal{M}$ of T^l, the loops of $\mathcal{M}$ are denoted by $\ell(\mathcal{M})$
36. Let $d_X(\mathcal{M}, T)$ denote the determinant of the submatrix of $A_X(T)$ formed by selecting the columns and rows associated with the loops of $\mathcal{M}$.
37. [3] $K(G) \cong \mathbb{Z}_{\Delta_1} \oplus \mathbb{Z}_{\frac{\Delta_2}{\Delta_1}} \oplus \cdots \oplus \mathbb{Z}_{\frac{\Delta_n}{\Delta_{n-1}}}$, where $\Delta_k = \gcd\left(\{d_X(\mathcal{M})|_{X=c} : \mathcal{M} \text{ is minimal 2-matching of } T^l\}\right)$ and $c = \mathrm{diag}(c(F_1), \ldots, c(F_s))$.

Bibliography

[1] Fleischner, Geller and Harary (1974): Outerplanar graphs and weak duals.

[2] Corrales and Valencia (2015): Critical ideals of trees.

[3] A. and Villagrán (2020): The structure of sandpile groups of outerplanar graphs.