# A Spectral Theory for Combinatorial Dynamics

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## We're gonna need a bigger vector space...

Given a set $X$, a map $T$ from $X$ to itself satisfying $T^n = \mathrm{Id}_X$ with $n \geq 1$, and functions $g_1, \ldots, g_k$ from $X$ to $\mathbb{C}$, define the function space

$$V = \langle g_i \circ T^j \mid 1 \leq i \leq k,\ 0 \leq j < n \rangle.$$

For all $f \in V$ we define the time-evolution operator $U : V \to V$ by

$$(Uf)(x) = f(T(x)).$$

$V$ is the smallest space of functions from $X$ to $\mathbb{C}$ that contains the linear span of $g_1, \ldots, g_k$ and is closed under $U$.

$V$ is at most $kn$-dimensional.

## ... and then we're going to chop it into pieces

$U$ is a diagonalizable operator on $V$ whose eigenvalues are $n$th roots of unity.

Let $V_\lambda$ be the span of the $\lambda$-eigenvectors, so that $V_1$ is the span of the fixed vectors, and let $V_1^\perp$ be the span of the other (non-unital) eigenvectors, so that $V = V_1 \oplus V_1^\perp$ and dim $V =$ dim $V_1 +$ dim $V_1^\perp$.

$V_1 = \mathrm{Ker}\,(I - U) = \mathrm{Im}\,(I + U + U^2 + \ldots + U^{n-1}) =$ the space of invariants.

$V_1^\perp = \mathrm{Im}\,(I - U) = \mathrm{Ker}\,(I + U + U^2 + \ldots + U^{n-1}) =$ the space of 0-mesies.

For every $m \geq 1$, the sum of the multiplicities of the eigenvalues $\zeta$ satisfying $\zeta^m = 1$ is the dimension of the space of invariants of $T^m$.

E.g., if $m = 1$, we get the space of invariants of $T$; if $m = n$, we get all of $V$.

## Putting the theory to use

Linear algebra packages that compute the ranks of matrices can be used to count the number of linearly independent invariants and the number of linearly independent 0-mesies.

Q. "Don't we need a basis for $V$ to use this?"
A. "Not as long as $X$ is finite."

We create an $|X|$-by-$kn$ matrix $M$ whose rows correspond to elements $x \in X$ and whose columns from left to right correspond to the respective functions

$$g_1, \ldots, g_k, Ug_1, \ldots, Ug_k, \ldots, U^{n-1}g_1, \ldots, U^{n-1}g_k,$$

where the entry in the $x$ row and the $U^j g_i$ column is $(U^j g_i)(x) = g_i(T^j x)$.

We call $M$ the *presenting matrix* of $U$. Its column-span is $V$.

dim $V =$ the rank of $M$

dim $V_1 =$ the dimension of the space of invariants = the rank of the matrix $M + M' + \ldots$ obtained by repeatedly cyclically shifting the columns

dim $V_1^\perp =$ the dimension of the space of 0-mesies = the rank of $M - M'$

## Example: Rotation of multisets in a cyclic group

Let $X_{n,k} =$ the set of weakly increasing $k$-tuples of elements of $\{0, 1, \ldots, n-1\}$.

Given $x \in X_{n,k}$, let $T(x)$ be the weakly increasing $k$-tuple obtained by incrementing each entry mod $n$ and then sorting.

For $1 \leq i \leq k$, let $g_i(x)$ be the $i$th element of $x$; that is, $g_i(x_1, x_2, \ldots, x_k) = x_i$.

E.g., with $n = 2$, $k = 3$:

$$\begin{aligned} T &: (0,0,0) \mapsto (1,1,1) \\ T &: (0,0,1) \mapsto (0,1,1) \\ T &: (0,1,1) \mapsto (0,0,1) \\ T &: (1,1,1) \mapsto (0,0,0) \end{aligned}$$

The presenting matrix is

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The shifted presenting matrix is

$$M' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

We have rank $M = 4$, rank $M + M' = 2$, and rank $M - M' = 2$; so there are 2 linearly independent invariants and 2 linearly independent 0-mesies.

When $n = 2$, the multiplicities of the eigenvalues 1 and $-1$ in the spectrum are $\lceil (k+1)/2 \rceil$ and $\lfloor (k+1)/2 \rfloor$ respectively; when $n > 2$, the multiplicity of the eigenvalue $\zeta$ in the spectrum is $\lfloor k/2 \rfloor + 1$ when $\zeta = 1$ and $k$ when $\zeta \neq 1$.

The proof uses (rigorous) calculations of matrix-ranks.

## Example: Rowmotion

For PL rowmotion on $J([2] \times [2])$, $V$ is 5-dimensional; $-1$ has spectral multiplicity 2 and the other 4th roots of 1 have multiplicity 1.

Work on $J([a] \times [b])$ is ongoing but stalled.

## Example: Lyness 5-cycle

For the PL Lyness 5-cycle, each 5th root of 1 has multiplicity 1.

Einstein has worked out the story for frieze patterns; other cluster algebras?

Preprint at http://jamespropp.org/spectrum.pdf

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