Moonshine

Mathieu Moonshine

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Monster Moonshine

Moonshine connects two a priori distinct areas of mathematics, Number theory and Group theory. The Klein J-function has the following expansion

$$J(q) = \frac{1}{q} + 196884q + 21493760q^2 + \dots$$

The Klein J-function is invariant under modular transformations (a modular form of weight 0). In 1978 the mathematician John McKay observed that the coefficients appearing in this expansion are the sums of dimensions of irreducible representations of the largest of 26 sporadic groups, the Monster group of order 8×10^{53}. This connection can be explained via string theory.

Modular Functions

A modular function of weight k, has the following transformation property:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Jacobi form of weight k and index m:

$$\phi_{k,m}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi imcz^2/(c\tau+d)}\phi_{k,m}(\tau,z)$$

$$\phi_{k,m}(\tau, z+\lambda\tau+\mu) = (-1)^{2m(\lambda+\mu)} e^{-2\pi im(\lambda^2+2\lambda z)}\phi_{k,m}(\tau,z) \quad \lambda,\mu \in \mathbb{Z}$$

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The elliptic genus

For superconformal field theories (SCFT) with more than $\mathcal{N} = (2,2)$ supersymmetry the elliptic genus is defined

$$Z^{ell}(q,y) = Tr_{RR}\left((-1)^{F_L+F_R} y^{J_0} q^{L_0-\frac{c}{24}} \overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\right)$$

where $y := e^{2\pi iz}, q := e^{2\pi i\tau}$, the right moving part is the Witten-index and therefore there is no dependence on $\overline{q}$.
For the $K3$ surface the elliptic genus is a Jacobi form of weight 0 and index 1.The elliptic genus is an index that counts BPS states of the theory and is independent of moduli of the theory.

$$Z^{ell}_{K3}(q,y) = 8\sum_{i=2}^{4}\left(\frac{\theta_i(q,y)}{\theta_i(q,1)}\right)^2$$

$$Z^{ell}_{K3}(q,y) = (massless\ characters) + \sum A_n ch_n(q,y)$$

$$\begin{aligned} A_1 &= 45 + \underline{45} \\ A_2 &= 231 + \underline{231} \\ A_3 &= 770 + \underline{770} \\ A_4 &= 2277 + \underline{2277} \\ &\dots \end{aligned}$$

In their paper from 2010 Eguchi, Ooguri and Tachikawa discovered a new Moonshine phenomena. They observed that $A_1, A_2, A_3, \dots$ are the sums of irreducible representations of the largest Mathieu group $\mathbb{M}_{24}$.

Heterotic String theory

Consider heterotic $E_8 \times E_8$ string theory compactified on $K3 \times T^2$.
We have to embed 24 instantons in oder to fulfill the Bianchi identity for $H3$, hence there are 13 different cases $(12-n, 12+n)$. The prepotential for the vector model for these cases is the same

$$F = STU + \frac{1}{3}U^3 + \frac{1}{(2\pi i)^3}\sum_{k>0,l\in\mathbb{Z};k=0,l>0} c(kl) L_{i3}\left(q_T^k q_U^l\right)$$

where S is the axion dilaton and T and U control the size and the complex structure of the torus.
The coefficients are directly related to the Eisenstein series

$$\frac{E_4(q)E_6(q)}{\eta(q)^{24}} = \sum_{m\geq -1} c(m)q^m = \frac{1}{q} - 240 - 141444q - \dots$$

The Eisenstein series $Ei(q)$ is a modular forms of weight i.
$E_6(q)$ directly relates to the elliptic genus of $K3$ at special points,

$$\frac{-4E_6(q)}{\eta(q)^{12}} = \left(\frac{\theta_2(q)}{\eta(q)}\right)^6 Z^{ell}_{K3}(q,-1) + \left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}} Z^{ell}_{K3}(q,-q^{-\frac{1}{2}}) - \left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}} Z^{ell}_{K3}(q,q^{\frac{1}{2}}).$$

Hence the prepotential directly relates to Z^{ell}_{K3} and hence to $\mathbb{M}_{24}$.

CHL orbifolds

We study CHL orbifolds, here the heterotic theory is orbifolded in such a way that supersymmetry is not broken. Hence we have to consider twined and twisted twined elliptic genera. We look at the McKay-Thompson series. In order to calculate the prepotential on the type IIA side we use another string duality, mirror symmetry.
In our paper arXiv: hep-th/1811.11619 and arXiv: hep-th/1911.09697 we give explicitly dual II A models for heterotic string theory on $K3 \times T^2/\mathbb{Z}_2$ and $K3 \times T^2/\mathbb{Z}_3$.

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