Model Theory, Quantifier Elimination and Differential Algebra II

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Outline

- A test for quantifier elimination.
- differentially closed fields (DCF)
- Quantifier Elimination for DCF and applications
- Canonical Definitions
- Further interplay between model theory and differential algebra (if time permits)

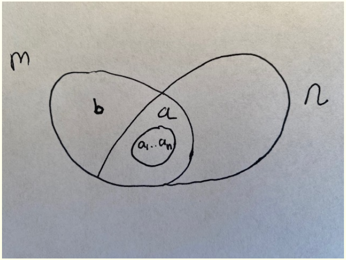
Model Theoretic test for QE

Theorem

Let T be a theory. Suppose that for all quantifier free formulas $\phi(x_1, y_1, \ldots, y_m)$, all $\mathcal{M}, \mathcal{N} \models T$, all $\mathcal{A} \subset \mathcal{M}, \mathcal{N}$ and all $a_1, \ldots, a_m \in \mathcal{A}$

(*) if $\mathcal{M} \models \exists x\ \phi(x, a_1, \ldots, a_n)$, then $\mathcal{N} \models \exists x\ \phi(x, a_1, \ldots, a_n)$.

Then T has quantifier elimination



QE for Algebraically Closed Fields

Let ACF be the axioms for algebraically closed fields

Theorem (Tarski)

ACF has quantifier elimination.

Suppose K, L are algebraically closed fields and $\mathcal{A} \subset K \cap L$ is a domain.
$\phi(v)$ is a quantifier free formula with parameters from $\mathcal{A}$ such that there is $b \in K$ with $K \models \phi(b)$.
$\phi(v)$ is a Boolean combination of formulas of the form $p(v) = 0$ where $p(X) \in \mathcal{A}[X]$.
Without loss of generality $\phi(v)$ is

$$\bigwedge_{i=1}^{n} f_i(v) = 0 \wedge g(v) \neq 0$$

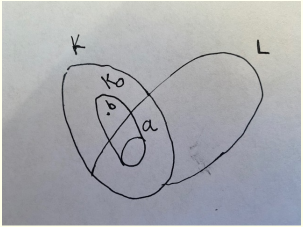
where $f_1, \ldots, f_n, g \in \mathcal{A}[X]$

QE for Algebraically Closed Fields

$$\bigwedge_{i=0}^{n} f_i(v) = 0 \wedge g(v) \neq 0$$

case 1 There are no nonzero f_i, in this case $\phi(v)$ is just $g(v) \neq 0$. We can find $c \in L$ such that $g(c) \neq 0$.

case 2 For some i, f_i is nonzero and $f_i(b) = 0$.
Let K_0 be the algebraic closure of $\mathcal{A}$ in K. Then $b \in K_0$



There is a field embedding $\sigma : K_0 \to L$ fixing $\mathcal{A}$ and $L \models \phi(\sigma(b))$.

Consequences of QE for ACF

- Let $\mathrm{ACF}_0 = \mathrm{ACF} \cup \{1+1 \neq 0, 1+1+1 \neq 0, \ldots\}$. Then ACF_0 axiomatizes $\mathrm{Th}(\mathbb{C})$. In particular, $\mathrm{Th}(\mathbb{C})$ is decidable.
- definable sets= quantifier free definable sets= Boolean combinations of algebraic varieties (i.e., *constructible sets*).
- (Chevalley's Theorem) The image of a constructible set under a polynomial map is constructible.

Existentially Closed Fields

We say that a field K is *existentially closed* if for any for any $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ and $L \supseteq K$ if the system

$$f_1(\mathrm{X}) = \cdots = f_m(\mathrm{X}) = 0$$

is solvable in L, then it is already solvable in K.

Corollary

Algebraically closed fields are exactly the existentially closed fields.

Every existentially closed field is algebraically closed.

Suppose $K \models$ACF and $K \subseteq L$ and $L \models \exists \mathrm{x}\ f_1(\mathrm{x}) = \cdots = f_m(\mathrm{x}) = 0$.

By quantifier elimination there is a quantifier free formula ψ (with parameters from K) equivalent to $\exists \mathrm{x}\ f_1(\mathrm{x}) = \cdots = f_m(\mathrm{x}) = 0$.

But $K \models \psi \Leftrightarrow L \models \psi$. Thus

$$K \models \exists \mathrm{x}\ f_1(\mathrm{x}) = \cdots = f_m(\mathrm{x}) = 0.$$

Differential Fields

A differential field (K, D) is a field K of characteristic 0 and a derivation $D : K \to K$.

$$D(x + y) = D(x) + D(y) \text{ and } D(xy) = xD(y) + yD(x).$$

We often write x' for $D(x)$.

The ring of *differential polynomials* $K\{X_1, \ldots, X_n\}$ is the polynomial ring

$$K[X_1, \ldots, X_n, X_1', \ldots, X_n', \ldots, X_1^{(m)}, \ldots, X_n^{(m)}, \ldots]$$

where we extend the derivation so that $D(X_i^j) = X_i^{j+1}$.

Question What are the existentially closed differential fields? Can we axiomatize this class?

Theorem

(A. Robinson) The existentially closed differential fields can be axiomatized.

Differentially Closed Fields

A *differentially closed field* is a differential field (K, D) where

- K is an algebraically closed field of characteristic 0;
- If $f, g \in K\{X\}$ are nonzero and $\mathrm{ord}(f) > \mathrm{ord}(g)$, then there is $x \in K$ such that $f(x) = 0$ and $g(x) \neq 0$.

We can give axioms DCF for this class.

Embarrassing Question What's an example of a differentially closed field?

No known natural examples.

Existence of Differentially Closed Fields

Lemma

If K is a differential field $f, g \in K\{X\}$ and $\mathrm{ord}(f) > \mathrm{ord}(g)$, there is a differential field $L \supset K$ and $a \in L$ such that $f(a) = 0$ and $g(a) \neq 0$.

Fact There is a prime differential ideal P with $f \in P$ such that if $h \in P$ $\mathrm{ord}(h) \geq \mathrm{ord}(f)$.

Let L be the fraction field of $K\{X\}/P$, then $K \subseteq L$.
Let $x = X/P$, then $f(x) = 0$, $g(x) \neq 0$.

Corollary

If K is a differential field there is $L \supseteq K$ with L differentially closed

Theorem

(L. Blum) DCF has quantifier elimination

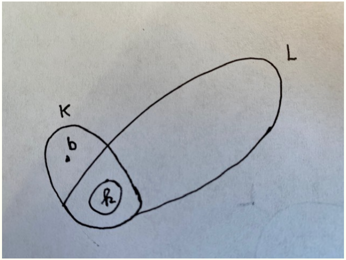
Let $K, L \models$DCF and let $k \subset K \cap L$ be a differential subfield and $b \in K \setminus k$.

If b satisfies $f_1(X) = \cdots = f_m(X) = 0 \wedge g(X) \neq 0$ where $f_i, g \in K\{X\}$ we need to find $c \in L$ satisfying the same.

It would suffice to find a differential field embedding $\sigma : k\langle b\rangle \to L$.

Without loss of generality we may assume that L is "very rich", namely if Σ is a set of equations and inequation from $k\{X\}$ and every finite subset of Σ is solvable in L then Σ is satisfiable in L.

Quantifier Elimination for DCF



We need to find $c \in L$ such that $f(b) = 0 \Leftrightarrow f(c) = 0$ for all $f \in k\{X\}$.

case 1 b is differentially transcendental over k

Let $\Sigma = \{g(X) \neq 0 : g \in k\{X\} \setminus \{0\}\}$.
For any nonzero $g_1, \ldots, g_n \in k\{X\}$ there is $c \in L$ such that $\prod g_i(c) \neq 0$. Thus every finite subset of Σ is solvable in L.
Thus, by richness, there is $c \in L$ differentially transcendental over k. Let $b \mapsto c$.

Quantifier Elimination for DCF

case 2 b is differentially algebraic over k.
Let $P = \{h \in k\{X\} : h(b) = 0\}$ a differential prime ideal in $k\{X\}$.

Fact Let $f \in P$ be of minimal order and degree.
Then P is the unique prime ideal over $k\{X\}$ such that $f \in P$ and no g of lower order is in P.

Let $\Sigma = \{f(X) = 0\} \cup \{g(X) \neq 0 : g \in K\{X\}, \mathrm{ord}(f) > \mathrm{ord} g\}$.
Since L is differentially closed any finite subset is solvable in L, thus Σ is solvable in L.
Let c be a solution to Σ then we can let $b \mapsto c$.

Consequences of Quantifier Elimination

- If $K, L \models DCF$ and $K \models \phi$, then $L \models \phi$. Find an equivalent quantifier free formula ψ then

$$K \models \psi \Leftrightarrow \mathbb{Q} \models \psi \Leftrightarrow K \models \psi$$

  So $K \models \phi \Leftrightarrow L \models \phi$.
- Definable sets are finite Boolean combinations of differential algebraic varieties.
- Differentially closed fields are existentially closed.
  Suppose $K \subseteq L$, $f_1, \ldots, f_m \in K\{X_1, \ldots, X_n\}$ and
  $L \models \exists x\ f_1(x) = \cdots = f_m(x) = 0$.
  There is a quantifier free formula ψ equivalent to
  $\exists x\ f_1(x) = \cdots = f_m(x) = 0$.
  $L \models \psi \Rightarrow K \models \psi \Rightarrow K \models \exists x\ f_1(x) = \cdots = f_m(x) = 0$.

Corollary

Suppose $P \subset K\{X_1, \ldots, X_n\}$ is a differential prime ideal and $g \notin P$. There is $x \in K^n$ such that $f(x) = 0$ for $x \in P$ and $g(x) \neq 0$.

Fact There are $f_1, \ldots, f_m \in P$ such that

$$\bigwedge_{i=1}^{m} f_i(\mathrm{x}) = 0 \Rightarrow f(\mathrm{x}) = 0 \text{ for all } f \in P.$$

Let L be the fraction field of $K\{\mathrm{X}\}/P$. In L let $a_i = X_i/P$. Then $f_1(\mathrm{a}) = \ldots f_m(\mathrm{a}) = 0 \wedge g(\mathrm{a}) \neq 0$.
By existential closedness we can find $\mathrm{a} \in K$.

Corollary

For any d, m, n *there is* k *(depending only on* d, m, n*) such that in any algebraically closed field* K *if* $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$ *have degree at most* d*, then* $f_1(\mathrm{X}) = \cdots = f_m(\mathrm{X}) = 0$ *has a solution in* K *if and only if*

$$1 \neq \sum_{i=1}^{m} g_i f_i$$

where each g_i *has degree at most* k*.*

Proof of Bounds

Write down generic polynomials $F_1, \ldots, F_m$ of degree d
i.e. $F_i = \sum_{|j| \leq d} c_{i,j} X^j$ (j a multi-index, $c_{i,j}$ new variables)
For each l there is a sentence Φ_l saying that

$$1 \neq \sum_{i \neq 1}^{m} g_i F_i$$

where each g_i has degree at most k.

Let T be the theory
$ACF \cup \{\forall x \neg \bigwedge_{i=1}^{n} F_i(x) = 0\} \cup \{\Phi_k : k = 1, 2, \ldots\}$.

T is not satisfiable. If we had a model of T, we would have a contradiction to Hilbert's Nullstellensatz.

By the Compactness Theorem. Some finite subset of T is not satisfiable.
But then there is a k such that if $F_1 = \cdots = F_m = 0$ has no solution, then we can find 1 using polynomials of degree at most k.

Existence of Canonical Parameters (Elimination of Imaginaries)

Fix K a differentially closed field.

(Definable Family) Suppose $A \subseteq K^{n+m}$ is definable and $B \subseteq K^m$ is definable. For $\mathrm{b} \in B$. We consider the family of definable sets $(A_{\mathrm{b}} : \mathrm{b} \in B)$ where

$$A_{\mathrm{b}} = \{\mathrm{a} \in K^n : \mathrm{b} \in B\}.$$

For example let A be the set of all $(x_1, x_2, y_1, \ldots, y_4) \in K^6$ such that

$$y_1 x_1 + y_2 x_2 D(x_1) = 0 \wedge y_2 D^{(2)}(x_2) - y_4 x_1 x_2 = 0$$

and let $B = C^4$ where C is the subfield of constants $C = \{x : D(x) = 0\}$.

Then for constants a, b, c, d

$$A_{a,b,c,d} = ax_1 + bx_2 D(x_1) = 0 \wedge cD^{(2)}(x_2) - dx_1 x_2 = 0$$

Existence of Canonical Parameters

Identification Problem Suppose X is in the family $(A_{\mathrm{b}} : b \in B)$. Can we determine for which b is $X = A_{\mathrm{b}}$?

i) Is there a unique $\overline{b}$ with $A_{\overline{b}} = X$? (Maybe not)

ii) If so and we know enough elements of X can determine $\overline{b}$?

Theorem (Poizat)

Suppose we have a definable family given by $A \subseteq K^{n+m}$ and $B \subseteq K^m$. Then there is a definable family given by $D \subseteq K^{n+l}$ and $E \subseteq K^l$ such that:

i) for all $\mathrm{b} \in B$, there is $\mathrm{e} \in E$ such that $A_{\mathrm{b}} = D_{\mathrm{e}}$;

ii) for all $\mathrm{e} \in E$, there is $\mathrm{b} \in B$ such that $A_{\mathrm{b}} = D_{\mathrm{e}}$;

iii) for all $\mathrm{e}_1, \mathrm{e}_2 \in E$ if $D_{\mathrm{e}_1} = D_{\mathrm{e}_2}$, then $\mathrm{e}_1 = \mathrm{e}_2$.

If $X = D_{\mathrm{e}}$, D_{e} a *canonical definition* of X and we call e a *canonical parameter*.

Finding the Canonical Parameter

Suppose D_{e} is a canonical definition of X.

Fact: Given sufficiently many "independent" $\mathrm{x}_1, \ldots, \mathrm{x}_N \in X$, we can express e is a differential rational function in $\mathrm{x}_1, \ldots, \mathrm{x}_{\mathrm{N}}$.

If D is "finite dimensional", we can bound N by $\dim(D) + 1$.

In applications often $D \subset C^m$ and $m + 1$ works as a bound.

Recently Ovchinnikov, Pillay, Pogudin and Scanlon, informed by the model theory, investigated identifiability problems much more concretely. Providing for example, reasonable algorithms to pass from a definable family to canonical definitions.

Theorem (Blum)

The theory of DCF is ω-stable.

There are powerful theories of dimension and independence that can be applied when studying differential algebra.

- (Existence and uniqueness of differential closures) For any differential field k there is a differentially closed field $K \supseteq k$ such that if $k \subset L$ is a differentially closed field there is a differential embedding of K into L fixing k. Moreover, K is unique up to isomorphism over k.

Interplay

- Differential Galois theory (Poizat, Pillay)
- Differential algebraic groups (Pillay)
- Diophantine applications (Hrushovski, Casale-Freitag-Nagloo)
- Differential algebraic geometry

Key Model Theoretic Problem about DCF

We say that a definable set $X \subset K^n$ is *strongly minimal* if for every definable $Y \subseteq X$ one of Y and $X \setminus Y$ is finite.

Problem Understand the strongly minimal sets.

- The constant field C, an algebraically closed field with no extra structure.
- Manin kernels–sets with only a divisible abelian group structure
  Start with A a simple abelian variety and let $A^\sharp$ be the closure of the torsion in the Kolchin topology.
- $y' = y^3 - y^2$ or generic Painlevé equations–sets with no structure

Thank You.

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