# A model-theoretic analysis of geodesic equations in negative curvature

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Model Theory of Differential Equations, Algebraic Geometry, and their Applications to Modeling

## Definition

A strongly minimal set $D$ is an infinite definable set such that every definable subset of $D$ (with parameters) is either finite or cofinite.

**Examples:**

- Any differential equation of the form

$$y' = f(y) \text{ with } f(X) \in \mathbb{C}(X)$$

  as the set of initial conditions itself is one dimensional.
- Certain higher-dimensional differential equations for example of the form

$$(E) : \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \quad \text{with } f, g \in \mathbb{C}[x,y]$$

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**A concrete consequence:** If $(E)$ is strongly minimal there are no rational change of coordinates $u = \phi(x,y), v = \psi(x,y)$ such that

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \longrightarrow \begin{cases} F(u,v,v') = 0 \\ G(u,u') = 0 \end{cases}$$

We fix once for all a rich differentially closed field $\mathcal{U}$ and we identify a differential equation $(E)$ with the associated definable set $D$.

(1) *The internal cover of the constants*: algebraic differential equations $(E)$ which (after reduction by the rational integrals) admit **a differential Galois theory in the sense of Kolchin**.

**Example**

Linear (possibly inhomogeneous) differential equations, Ricatti equations, elliptic differential equations and higher dimensional variants.

In that case, the definable set $D$ is part of a principal definable homogeneous space for an algebraic group $G$ over the field of constants.

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(2) *Purely locally modular strongly minimal sets*: They are principal homogeneous spaces for the action of a strongly minimal modular group.

- The strongly minimal modular groups of $\mathbf{DCF}_0$ are called **Manin Kernels**.
- They have been entirely classified in the 90's by Hriushovski and Pillay.
- The simplest examples are 2-dimensional non autonomous differential equations (part of a 3-dimensional autonomous equation).

(3) *Disintegrated strongly minimal set.* constituted of strongly minimal sets $D$ with the most degenerated structures (for instance, $D$ do not interpret an infinite definable group).

**Example:** The first Painlevé equation $y'' = 6y^2 + t$.
We will think about the solutions of such equation as "highly transcendental" (or new transcendental) functions (terminology justified later).

# Zilber's trichotomy in differentially closed fields II

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**Theorem (Semi-minimal analysis of algebraic differential equations)**

*Let $(E)$ be any algebraic differential equation over $(K, \delta)$ and $\mathcal{U}$ a rich differentially closed field.*
*There exists a sequence $a_0, \ldots, a_n$ of tuples from $\mathcal{U}$ such that for every $i < n$*

(o) *either $a_{i+1}$ is a constant;*

(i) *or $a_{i+1}$ satisfy a differential equation over $K(a_0, \ldots, a_i)^{alg}$ which admit a Galois theory in the sense of Kolchin;*

(ii) *or $a_{i+1}$ is a solution of a principal homogeneous space for a Manin Kernel over $K(a_0, \ldots, a_i)^{alg}$;*

(iii) *or $a_{i+1}$ satisfies a minimal disintegrated differential equation over $K(a_0, \ldots a_i)^{alg}$.*

*and $(E)$ admits a generic solution in $K(a_0, \ldots, a_n)^{alg}$.*

We consider a Hamiltonian system in symplectic coordinates $(p, q)$ of the form:

$$H(p, q) = \frac{1}{2}p^2 + V(q) \text{ with } V(q) \in \mathbb{C}[q]$$

- Since the Hamiltonian $H$ is always a rational integral, we are led to study a first-order differential equation after adding a new constant $c \notin \mathbb{C}$,

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- Using separation of variables, we obtain the following integral formulation:

$$(*) : t = \int \frac{dq}{\sqrt{2c - 2V(q)}}$$

- *Classical question:* With a change of variables, can one reduce the computation of $(*)$ to a rational or an elliptic integral?
  - if yes, then the Hamiltonian equation is solvable with operations (o) and (i) only. This is always the case when $deg(V) \leq 5$.
  - if no, then the Hamiltonian equation is solvable with the operations (o) and (iii). This is generically the case as soon as $deg(V) > 5$.

**Question**

*Is it possible to describe effectively the semi-minimal analysis of higher dimensional Hamiltonian systems appearing in classical mechanics?*

*An idea of my PhD:* the **geodesic differential equation of a compact Riemannian surface with negative curvature** is a good test problem:

- It is a non-completely integrable Hamiltionian system with two degrees of freedom, like many other interesting systems of classical mechanics.
- As shown by Anosov ('69), the dynamic of the vector field satisfies global hyperbolic properties. It makes it easier to study than other non-integrable Hamiltonian system which are closer to completely integrable ones.

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To study the semi-minimal analysis, we need to work with **algebraically presented Riemannian manifold** and to complexify the differential equation.

**Setting:** We start with $(X, g)$ a pseudo-Riemannian algebraic variety over $\mathbb{R}$ that is a smooth algebraic variety endowed with a non degenerate symmetric 2-form over $\mathbb{R}$ such that:

- $X(\mathbb{R}) \neq 0$ ($\Rightarrow$ Zariski-dense in $X$).
- $(X(\mathbb{R}), g_{\mathbb{R}})^{an}$ is a real-analytic Riemannian manifold (of dimension two) with negative (but in general **variable**) curvature.

Let $(M, g)$ be a compact Riemaniann manifold of dimension 2.

- The geodesic differential equations of $(M, g)$ is the Hamiltonian system on $TM$ associated with the "free" Hamiltonian:

$$H(x, y) = \frac{1}{2} g_x(y, y)$$

- We obtain a vector field $v_H$ on $TM$ such that the Hamiltonian defines a first integral:

$$H : TM \to \mathbb{A}^1.$$

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**Definition**

The (unitary) geodesic differential equation of $(M, g)$ is the differential equation of dimension three such that:

- The underlying manifold is the sphere bundle $SM \subset TM$ of $M$.
- It is given by the restriction to $SM$ of the vector field $v_H$ on $TM$.

We now assume that $(M, g)$ has everywhere negative (but in general non constant) curvature.

- *Global hyperbolic structure:* there is a decomposition

$$T_{SM} = E_s \oplus E_0 \oplus E_u$$

into $(d\phi_t)_{t\in\mathbb{R}}$-invariant continuous line bundles such that $E_0$ is the direction of the vector field, $(d\phi_t)_{t\in\mathbb{R}}$ is exponentially contracting on $E_s$ and exponentially expanding $E_u$.

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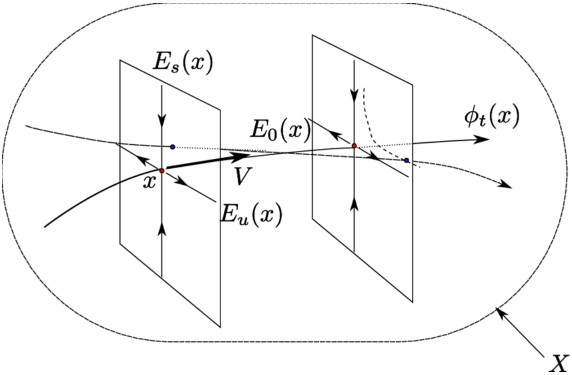
- *Periodic orbits*: Periodic points of $(\phi_t)_{t\in\mathbb{R}}$ are dense in $SM$.
- *Ergodic and mixing properties* The dynamic of $(SM, (\phi_t))_{t\in\mathbb{R}}$ is topologically (weakly) mixing that is:

$$(SM, (\phi_t)_{t\in\mathbb{R}}) \text{ and } (SM \times SM, (\phi_t \times \phi_t)_{t\in\mathbb{R}}) \text{ both admit a dense orbit.}$$

This implies that for all $n \geq 3$, $(SM^n, (\phi_t \times \ldots \times \phi_t)_{t\in\mathbb{R}})$ admits a dense orbit too.

# A local picture of an Anosov flow of dimension three




We will compute in the frame given by $(E_s(x), E_0(x), E_u(x))$.
Even when the inital data is real analytic or algebraic, in general, $E_s(x)$ and $E_u(x)$ only depends continuously on $x$.

**Theorem**

*Let $(X, g)$ be a pseudo-Riemannian variety over $\mathbb{R}$ such that the real-analytification $(X(\mathbb{R}), g_{\mathbb{R}})^{an}$ is a compact (non-empty) connected Riemannian surface with negative curvature.*
*Consider the geodesic differential equation $(SX, v)$ of $(X, g)$ and denote by $D$ the associated definable set.*

(1) *The generic type of $D$ is minimal and disintegrated.*
(2) *In other words, the generic solution of $(SX, v)$ does not lie in a differential field of the form*

$$(\mathbb{C}(t), \frac{d}{dt}) = (K_0, \delta_0) \subset (K_1, \delta_1) \subset \ldots \subset (K_n, \delta_n).$$

*where each elementary step $(K_i, \delta_i) \subset (K_{i+1}, \delta_{i+1})$ is either an algebraic extension or obtained by one of the operations $(o), (i), (ii)$ and:*

*$(iii)_3$: solving a strongly minimal disintegrated equation* ***living in dimension $< 3$****.*

# Comments on the non-generic behavior

- It is a generic statement. It is possible to show that the non generic behavior is concentrated on a subset $\mathcal{Z}_1 \subset SX$ which is an (at most) countable union of proper closed invariant subvarieties of $SX$.

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- The geodesic flow on a compact Riemannian is far for being uniquely ergodic: there are many non-trivial subflows (for instance at least all the periodic orbits).

To study the non-generic behavior, we are led to:

**Question:** Can a **compact** algebraically presented Riemannian manifold **of negative curvature** contain infinitely many algebraic geodesics?

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- **Disintegration property:** For generic solutions of $(SX, v)$

$$\text{pairwise algebraic indepedence} \Rightarrow \text{algebraic independence}$$

To describe pairwise independence, we are led to:

**Question:** Can a **compact** algebraically presented Riemannian manifold **of negative curvature** admit a finite non trivial (resp. infinite) group of algebraic isometries?

- **Geodesics on smooth quadrics**: Let $E = X(\mathbb{R})$ be an Euclidean ellipsoid. The geodesic flow is a completely integrable Hamiltonian system so we have a second (non-trivial) rational integral

$$H_2 : (SX, v) \to \mathbb{A}^1$$

The generic fibre of $H_2$ can be realized as a an invariant subvariety of an Abelian variety endowed with a translation invariant vector field (Jacobi, Moser).

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- **Geodesics on surfaces of revolution** It is also a completely integrable Hamiltonian system, so $p$ is not minimal.

In some cases, $p$ is analyzable in the constants, in others, one needs to solve a disintegrated differential equation of order one $f(y, y') = 0$.

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- **Geodesics on an almost sphere** $X(\mathbb{R}) = \mathbb{S}^2$ and $g = g_0 + \epsilon g_1$ a small non-integrable perturbation of the Euclidean metric on the sphere.

One expects a similar description than in negative curvature.

**Theorem**

*Let $p$ be a prime number. Assume that $(X, v)$ is a differential equation of dimension $p$ satisfying:*

(i) *The generic type of $(X, v)$ is orthogonal to the constants.*

(ii) *Every foliation $\mathcal{F}$ on $X$ of rank $r \in \{1, \ldots, p-1\}$ which is invariant under the vector field $v$ has a Zariski-dense leaf.*

(iii) *Every $p$-web $\mathcal{W}$ of foliations by curves on $X$ which is invariant under the vector field $v$ has a Zariski-dense leaf.*

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- In (ii), a foliation $\mathcal{F}$ of rank $r$ on $X$ is a saturated coherent subsheaf of rank $r$ of the locally-free sheaf $\Theta_{X/k}$ stable under Lie bracket. It is called invariant under $v$ if

$$[v, \mathcal{F}] \subset \mathcal{F}$$

- In (iii), a (generically smooth) $r$-web $W$ of foliations by curves on $X$ is a closed subvariety $W \subset \mathbb{P}(T_X)$ such that all irreducible components of $W$ dominate $X$ and $\pi_{|W} : W \to X$ is generically finite.
  The vector field $v$ has a first projective prolongation $\mathbb{P}(v)$ to $\mathbb{P}(T_X)$ from which derives the notion of invariance for webs.

Thank you for your attention!