Irreducibility and generic ODEs

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Introduction and Motivation

- Strong minimality is a central notion in model theory.

- The Key Model Theoretic Problem about DCF (D. Marker's talk):

  Understand the strongly minimal sets.

- This is a problem internal to Model theory/DCF (although it has been applied very successfully).

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- Strong minimality is a central notion in model theory.
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- This is a problem internal to Model theory/DCF (although it has been applied very successfully).
- **Goal:**
  1. Use the slightly more general notion irreducibility to explain the relevance outside model theory.
  2. Give an idea of the problem of proving that a differential equation is strongly minimal.

Irreducibility: All solutions are 'new'

Let us give the *Painlevé-Umemura definition* of a classical/known functions.

- In what follows we will identify a meromorphic function f on an open set $U \subset \mathbb{C}$ with its restriction $f_{|V}$ onto an open subset $V \subset U$.

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- In what follows we will identify a meromorphic function f on an open set $U \subset \mathbb{C}$ with its restriction $f_{|V}$ onto an open subset $V \subset U$.
- S will denote certain set of meromorphic functions on a domain $U \subset \mathbb{C}$.
- We assume that all the elements in S are already known functions.
- One then define the permissible operations to obtain other know functions from S.

(O) Let $f \in S$. Then f' is a known function.

(P1) Let $f_1, f_2 \in S$, then the functions $f_1 \pm f_2$, $f_1 \cdot f_2$ and f_1/f_2 (if $f_2 \neq 0$) are known functions.

(P2) If f is a solution of an equation $X^n + a_1 X^{n-1} + \ldots + a_n = 0$, with $a_i \in S$, then f is a known function.

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(P5) Let $\Gamma \subset \mathbb{C}^n$ be a lattice such that the quotient $\mathbb{C}^n/\Gamma$ is an abelian variety. Let $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Gamma$ be the projection. Let ϕ be a meromorphic function on $\mathbb{C}^n/\Gamma$. Then $\phi \cdot \pi \cdot (f_1, \ldots, f_n)$, where $f_1, \ldots, f_n \in S$, is a known function.

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For each $k \in \mathbb{N}_{>0}$

(Q_k) If f is a solution of an ODE $G(y, y', \ldots, y^{(k)}) = 0$ where G has coefficients from S. Then f a known function.

Here $F \in \mathbb{C}(t)(X_1, \dots, X_n)$ denotes a rational function and $n > 1$.

Definition

The equation $y^{(n)} = F(y, y', \dots, y^{(n-1)})$ is PU-irreducible if, starting from the set of constant functions $\mathbb{C}$, one cannot express any of its solutions by a finite iteration of the permissible operations (0), (P1), $\dots$, (P5) and $(Q_1), \dots, (Q_{n-1})$.

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1. It has no solution solution in $\mathbb{C}(t)^{alg}$; and
2. For any solution f and any finitely generated differential field extension K of $\mathbb{C}(t)$, either

$$f \in K^{alg} \qquad \text{or} \qquad tr.deg(K(f, f', \ldots, f^{(n-1)})/K) = n.$$

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The equation $y^{(n)} = F(y, y', \ldots, y^{(n-1)})$ is PU-irreducible if

1. It has no solution solution in $\mathbb{C}(t)^{alg}$; and
2. The set defined by the equation in a differentially closed field is strongly minimal.

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- It can be quite hard to prove that an ODE is strongly minimal.
- **Example:**

  The equation $y'' = 2y^3 + ty + \frac{1}{2}$ is not strongly minimal because of the existence of

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$$(y')^3 - (y^2 + \tfrac{t}{2})(y')^2 - (y^4 + ty^2 + 4y + \tfrac{t^2}{4})y' + y^6 + \tfrac{3}{2}ty^4 + 4y^3 + \tfrac{3}{4}t^2y^2 + 2ty + 2 + \tfrac{t^3}{8} = 0$$

- The equation $P_{II}(\alpha)$:

$$y'' = 2y^3 + ty + \alpha, \quad \alpha \in \mathbb{C}$$

  is the second Painlevé equation.

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- If w is a solution of $P_{II}(\alpha)$, then

$$\begin{aligned} T_+(w) &= -w - \frac{\alpha + 1/2}{w' + w^2 + t/2} \\ T_-(w) &= -w + \frac{\alpha - 1/2}{w' - w^2 - t/2} \end{aligned}$$

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Fact

The degree in y' of the order 1 subvariety of $P_{II}(1/2 + n)$, $n \in \mathbb{N}$, is 3^n.

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Result (Freitag-Jaoui-N)

The solution set of equation

$$y'' = y' \frac{p(y)}{q(y)} \tag{0.1}$$

where the rational function $\frac{p(y)}{q(y)} \in \mathbb{C}(y)$ *has a simple pole at* $y = 0$*, is strongly minimal (and so geometrically trivial).*

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- We can hence take a Puiseux series expansion of f'

$$u = \sum_{i=r}^{\infty} a_i \tau^i$$

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where $\tau^e = f$ for some $e \in \mathbb{N}_{>0}$ and $a_i \in K^{alg}$.

- Plug in the equation $(u)' = \frac{u}{f}$ and get a contradiction.

The Painlevé equation (again): $y'' = 2y^3 + ty + \alpha$

- Painlevé (1895) claimed that at least for generic values of the parameters all the Painlevé equations would be strongly minimal.
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- Genericity matters outside model theory:

  Peter J. Forrester and Nicholas S. Witte, Painlevé II in random matrix theory and related fields, Constr. Approx. 41 (2015), no. 3, 589-613.

Schwartzian triangle Equations

- The Schwartzian triangle equation $\mathcal{S}(\alpha, \beta, \gamma)$ is given by

$$S_t(y) = \frac{1}{2}\left(\frac{1-\beta^{-2}}{y^2} + \frac{1-\gamma^{-2}}{(y-1)^2} + \frac{\beta^{-2}+\gamma^{-2}-\alpha^{-2}-1}{y(y-1)}\right)(y')^2$$

- where

$$S_t(y) = \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2$$

- is the Schwartzian derivative and $\alpha, \beta, \gamma \in \mathbb{C}$.

- The solutions of $\mathcal{S}(\alpha, \beta, \gamma)$ are conformal mapping of hyperbolic triangle to the complex upper half plane.
- The solutions of $\mathcal{S}(k, l, m)$ when $2 \leq k \leq l \leq m$ (integers or ∞) and $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$ are Fuchsian automorphic functions.

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Result (Casale-Freitag-N and Freitag-Scanlon for $(2, 3, \infty)$)

The equation $\mathcal{S}(k, l, m)$, with $2 \leq k \leq l \leq m$ (integers or ∞) and $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1$, is strongly minimal (and much more...).

- We use the above result to prove a deep functional transcendence result called the Ax-Lindemann-Weierstrass theorem with derivative for the Fuchsian automorphic functions.

Result (Blázquez Sanz-Casale-Freitag-N)

If α, β, γ are algebraically independent over $\mathbb{Q}$, then the equation $\mathcal{S}(\alpha, \beta, \gamma)$ is strongly minimal (and much more. . .).

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- **Key method in the proof**:

  Here $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}^n$ are algebraically independent over $\mathbb{Q}$

Fact

Let $\theta(x_1, x_2, \ldots, x_n)$ be a formula in the language $(0, 1, +, \times, D)$ such that

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Then for all but finitely many $\hat{\alpha} \in \mathbb{C}$ we have

$$\mathcal{U} \models \theta(\hat{\alpha}, \alpha_2, \ldots, \alpha_n)$$

Fun Strategy

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  Step 2: Use Backlund transformations to deduce that $P_{II}(n)$ strongly minimal for all $n \in \mathbb{Z}$.

  Step 3: Conclude, using previous fact, that $P_{II}(\alpha)$ is strongly minimal for transcendental α.

Thank you very much for your attention.