Tropical algebraic geometry

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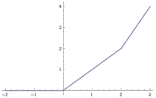
BIRS Workshop on Model Theory of Differential Equations, Algebraic Geometry, and their Applications to Modeling.

1-5 June 2020

What is tropical algebraic geometry?

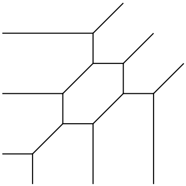
$f(x) = \max(2x - 2, x, 0)$



piecewise linear convex functions on $\mathbb{R}^n$

⇣ bend locus

piecewise linear structures in $\mathbb{R}^n$

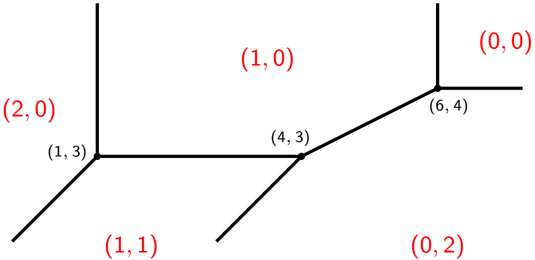


Tropical geometry of product mix auctions

$(0,0)$	$(1,0)$	$(0,1)$	$(2,0)$	$(1,1)$	$(0,2)$	bundles of goods
						$+$
0	6	4	7	9	8	agent evaluation

$$\max(0, 6-x, 4-y, 7-2x, 9-x-y, 8-2y)$$
$$= 0 \oplus 6x^{-1} \oplus 4y^{-1} \oplus 7x^{-2} \oplus 9x^{-1}y^{-1} \oplus 8y^{-2}$$

maximal profit

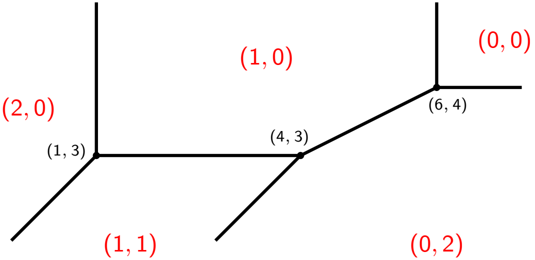


bundles maximizing profit

$+$

locus of indifference

Tropical geometry of product mix auctions



bundles minimizing loss

+

locus of indifference

Questions

- Under what circumstances are competitive equilibria guaranteed?
- How to check whether competitive equilibrium exists?

Answer [Baldwin-Klemperer 2019]

Look at the tropical hypersurfaces and their intersection patterns.

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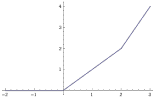


piecewise linear convex functions on $\mathbb{R}^n$

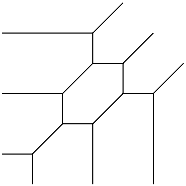
bend locus

polynomial structures in K^n ⇢ piecewise linear structures in $\mathbb{R}^n$

$z \longmapsto -\nu(z)$

valued field

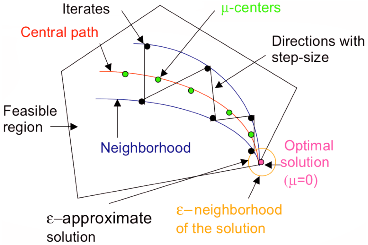
$y^2 - x^3 + x = 0$



Tropical geometry in linear optimization

Smale's 9th problem, continuous Hirsch conjecture

- Can linear programming be solved in strongly polynomial time?
- Is the curvature of the central path linear in the number of constraints?



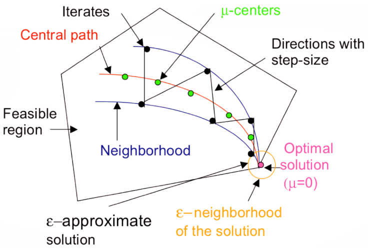
Tropical geometry in linear optimization

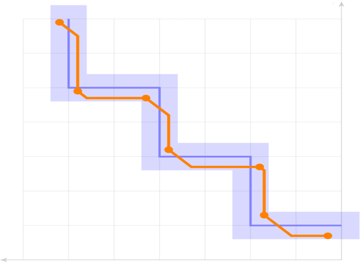
Smale's 9th problem, continuous Hirsch conjecture

- Can linear programming be solved in strongly polynomial time?
- Is the curvature of the central path linear in the number of constraints?

Partial answer [Allamigeon-Benchimol-Gaubert-Joswig 2019]

No, the curvature can be exponential.





$$\mathbb{R}\{\{t\}\}^n \overset{\nu}{\dashrightarrow} \mathbb{R}^n$$

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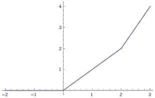

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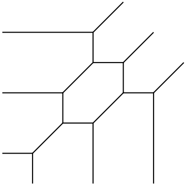

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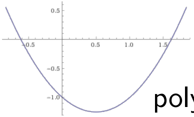

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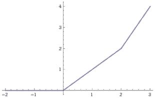

polynomials in $K[x_1, \ldots, x_n]$ - - - - - → piecewise linear convex functions on $\mathbb{R}^n$

zero locus ↓ Trop ↘ ↓ bend locus

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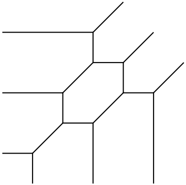
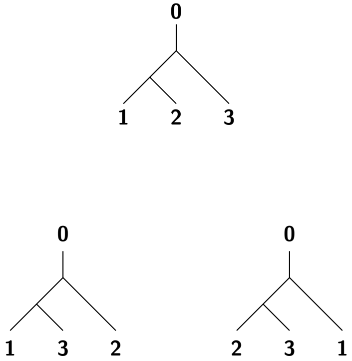


Tropical geometry in phylogenetics

Question

Phylogenetic trees are labelled metric trees. What shape has their space?



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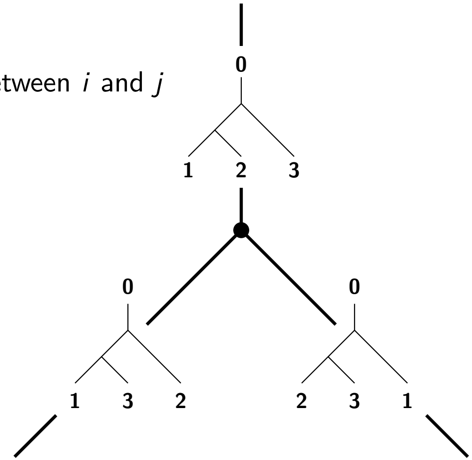
$p_{ij} :=$ distance between i and j



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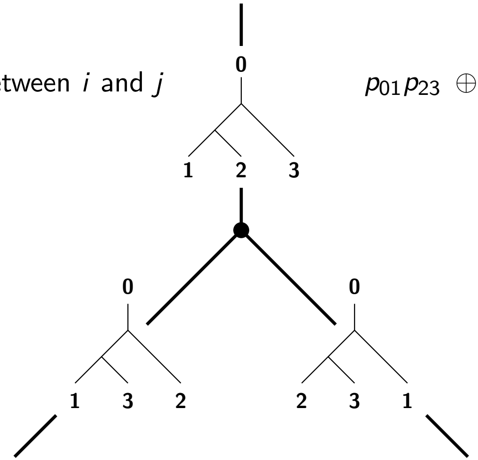
$p_{01}p_{23} \oplus p_{02}p_{13} \oplus p_{03}p_{12}$
attained twice



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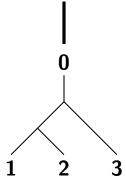

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Answer [Speyer-Sturmfels 2004]

Spaces of n-leaved phylogenetic trees are tropical Grassmannians TGr(2,n).

Tropical geometry in phylogenetics

Answer [Speyer-Sturmfels 2004]

Spaces of *n*-leaved phylogenetic trees are tropical Grassmannians TGr(2,n).

Idea [Yoshida et al 2017+]

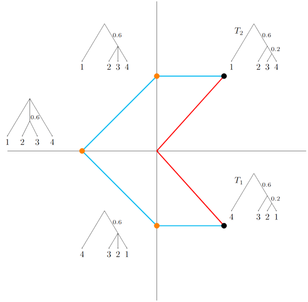
Data science on tropical Grassmannians instead of BHV spaces.

Advantages:

- Polish spaces,
- geodesic paths of smaller depths,
- geodesic triangles two-dimensional,
- less sticky means.

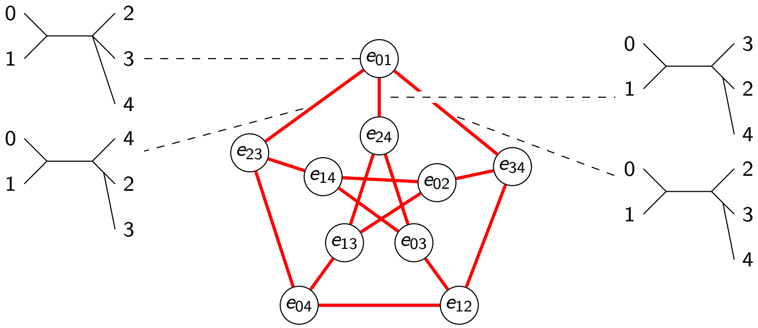
Disadvantages:

- geodesic paths not unique.



Tropical geometry in phylogenetics

The tropical Grassmannian TGr(2,5)



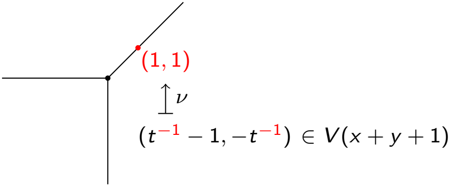
The Peterson-Graph

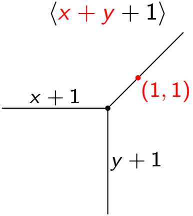
Structure and Fundamental Theorem

The tropical variety of $I \subseteq K[x^{\pm 1}]$ prime or $V(I) \subseteq (K^*)^n$ irreducible is the support of a balanced polyhedral complex of equal dimension, pure and connected in codimension one:

$$\overline{\{-\nu(z) \in \mathbb{R}^n \mid z \in V(I)\}} = \mathrm{Trop}(I) = \{w \in \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ mon. free}\}$$

$$\mathrm{Trop}(I) = \bigcap_{f \in I} \mathrm{Trop}(f)$$





Computational tools:

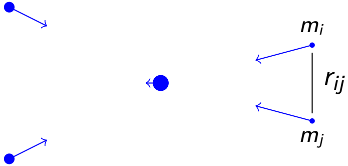
- Polynomial system solving
- Gröbner bases

Structure and Fundamental Theorem

The tropical variety of $I \subseteq K[x^{\pm 1}]$ prime or $V(I) \subseteq (K^*)^n$ irreducible is the support of a balanced polyhedral complex of equal dimension, pure and connected in codimension one:

(example.u3d)

Tropical geometry in celestial mechanics



central configurations in the N-body problem

Conjecture

Only finitely many up to symmetry.

Proof for $N = 4, 5$ [Hampton-Moeckel-Jensen 2006+2011]

$$I := \left\langle \sum_{k=1}^{N} m_k \left[(r_{ij} - 1)^{-3} (r_{jk}^2 - r_{ij}^2 - r_{ij}^2) + \ldots \right] \right\rangle \qquad \dim\Big(V(I) \Big) = 0$$

$$\Cup \qquad\qquad\qquad \Uparrow$$

$$f_1, \ldots, f_k \quad \text{with} \quad \dim\Big(\bigcap_{i=1}^{k} \mathrm{Trop}(f_i) \Big) = 0$$

$$\cup|$$

$$\mathrm{Trop}(V(I)) = \bigcap_{f \in I} \mathrm{Trop}(f)$$

Tropical enumerative geometry

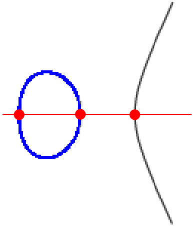
Question [Gromov-Witten invariants]

How many plane algebraic curves of degree d and genus g go through $3d + g - 1$ points in general position?

Answer [Mikhalkin 2005]

As many as there are tropical curves (counted with multiplicity).

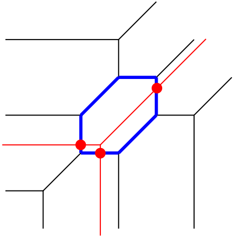
$V(I) := \{z \in K^n \mid f(z) = 0 \ \ \forall f \in I\}$ $\qquad$ $\text{Trop}(I) := \nu(V(I) \cap (K^*))$



dimension: 1

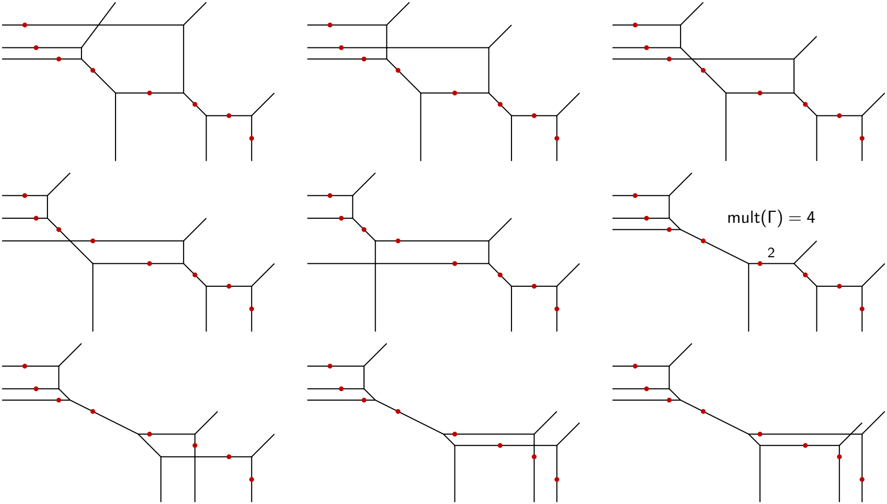
degree: 3

genus: 1



Tropical enumerative geometry

$d = 3,\ g = 0$:



Algebraic geometry vs tropical algebraic geometry

Algebraic varieties and tropical varieties are fascinatingly similar

- Gromov-Witten invariants
- Welchinger invariants
  [Itenberg-Kharlamov-Shustin 2009-2013], [Gathmann-Markwig-Schröter 2013]
- J-invariants [Katz-Markwig-Markwig 2009]
- Igusa invariants [Helminck 2016]
- Smooth non-hyperelliptic curves of genus 3 are plane quartics
  [Hahn-Markwig-R.-Tyomkin 2019]

Algebraic geometry vs tropical algebraic geometry

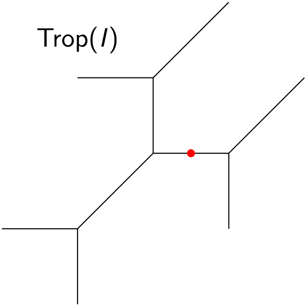
Algebraic varieties and tropical varieties are fascinatingly similar and yet inherently different:

- Tropical cubic surfaces contain infinitely many tropical lines [Panizzut-Vigeland 2019]

(example2.u3d)

- Tropical quartic curves have infinitely many bitangents [Baker-Len-Morrison-Pflueger-Ren 2016]
- Tropical sextic curves have infinitely many tritangents [Harris-Len 2017]

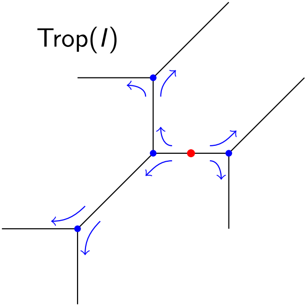
Computing tropical varieties



[Bogart-Jensen-Speyer-Sturmfels-Thomas 2007] [Maclagan-Chan 2018] [Hofmann-R. 2018] [Markwig-R. 2019]

1. Compute a starting point.

Computing tropical varieties



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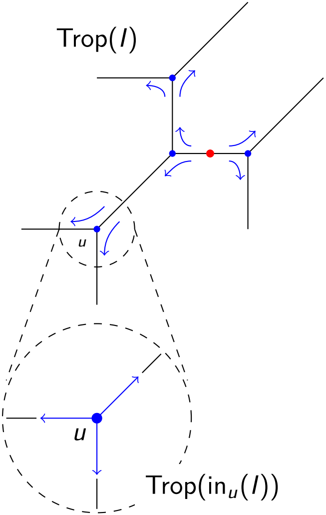

finite union of extremal rays

[Bogart-Jensen-Speyer-Sturmfels-Thomas 2007] [Maclagan-Chan 2018] [Hofmann-R. 2018] [Markwig-R. 2019]

1. Compute a starting point.
2. Compute the rest.

Relies theoretically on

- Fundamental Theorem
- Structure Theorem

Builds practically on

- convex hull algorithms
- (tropical) Gröbner bases
- (non-archimedian) polynomial system solving

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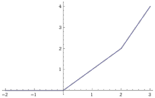

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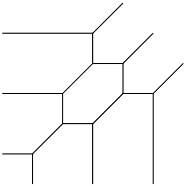

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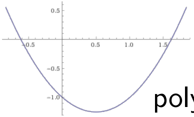

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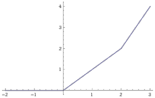

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zero locus ↓   Trop ↘   ↓ bend locus

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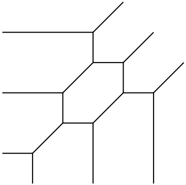

What is tropical differential geometry?

$z \mapsto -\nu(z)$

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differential polynomials in $R\{x_1, \ldots, x_n\}$ ⇢ trop. differential polynomials in $\overline{\mathbb{R}}\{x_1, \ldots, x_n\}$

solutions

Trop

bend locus

solution sets in R^n —supports⇢ solution supports in $\mathcal{P}(\mathbb{Z}^n_{>0})$

$1 + t^4 \mapsto \{0, 4\}$

diff. valued ring
e.g., $\mathbb{C}[\![t]\!]$

Fundamental theorem [Aroca-Garay-Toghani 2016]

The diagram commutes for differential ideals.

What is tropical differential geometry?

Motivation:

- Combinatorial approach to p-adic differential equations.
- Global solutions of chemical reaction networks.

Currently known:

- Fundamental Theorem [Aroca-Garay-Toghani 2016]
- Generalization of Fundamental Theorem to PDEs [Falkensteiner-Garay-Haiech-Noordman-Toghani-Boulier 2020]
- Tropical differential Gröbner bases [Hu-Gao 2019]

Currently unkown:

- Structure Theorem
- Finiteness of tropical differential bases

Read also:

- Cristhian Garay: *Using tropical differential equations* (March 2020).