Hilbert's 16th problem and o-minimality

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Are there other qualitative phenomena needed to describe the phase portrait of F?

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- **Dulac's problem:** if F extends to a real analytic vector field on the sphere S^2, then F has finitely many limit cycles.
- **Hilbert's 16th problem (H16):** if F is polynomial of degree d, there exists $H(d) \in \mathbb{N}$, depending only on d, such that F has at most $H(d)$ many limit cycles.

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In the case of Poincaré's example, the first return map is real analytic at 0, so there are only finitely many limit cycles near the cycle C.

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While Dulac completed Point 1, Point 2 was the gap left unproved by him and proved 70 years later by Ecalle and Ilyashenko.

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Good news: Roussarie shows that if all singularities of F_μ are isolated, then P is always a polycycle.

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Fact

There are open neighbourhoods U of μ and V of P such that the transition maps $f_{\mu',i}$ and $g_{\mu',i}$ are well defined for all parameters $\mu' \in U$ and segment coordinates $x_i, y_i \in V$.

definability

Now $x_1 \in I_1 \cap V$ corresponds to a cycle of $F_{\mu'}$ near P, with $\mu' \in U$, if and only if

- there exist $x_2, \ldots, x_{k+1}, y_1, \ldots, y_k$ such that $y_i = g_{\mu',i}(x_i)$ and $x_{i+1} = f_{\mu',i}(y_i)$ for $i = 1, \ldots, k$, and $x_{k+1} = x_1$.

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So FCC follows from (UF) and the following:

Conjecture (o-minimality)

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This would imply that Roussarie's FCC conjecture holds for all polycycles containing only hyperbolic singularities (a generic case of FCC).