Automorphisms of projective surfaces: finite orbits of large groups

Based on a joint work with Romain Dujardin

Dynamics on a real K3 surface (C.T. McMullen, V. Pit)

— I —

Automorphisms of surfaces:

Examples

- X = smooth complex projective surface (real dimension 4)
- $\mathrm{Aut}(X)$ = group of holomorphic diffeomorphisms
  = group of (regular, algebraic) automorphisms
  = a complex Lie group.

- **Example 1.–** $E = \mathbf{C}/\Lambda$, an elliptic curve.

  $X = E \times E = \mathbf{C}^2/(\Lambda \times \Lambda)$.

  X = translations $\subset \mathrm{Aut}(X)$.

  $\mathrm{GL}_2(\mathbf{Z}) \subset \mathrm{Aut}(X)$.

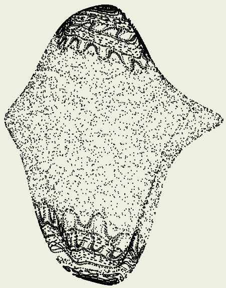
- **Example 2.–** $\eta(x, y) = (-x, -y)$ on $X = E \times E$.

  η commutes to the action of $\mathrm{GL}_2(\mathbf{Z})$.

  $Y = \widehat{X/\eta}$ is a **Kummer surface**.

- **Example 3.–** $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, smooth, degree $(2,2,2)$:

$$x^2y^2z^2 + (x^2y^2 + y^2z^2 + z^2x^2)/200 + x^2 + y^2 + z^2 + xy + z - y = 6.$$



Real orbits of a K3 automorphism.

Closer view from above.

— II —

Cohomology: Minkowski space and types of automorphisms.

- **Intersection form.–**

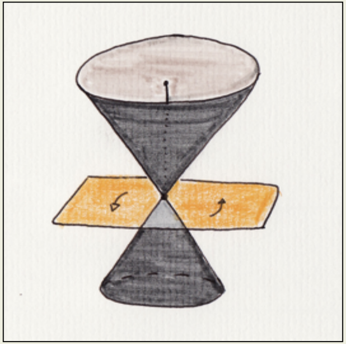
$$\begin{aligned} \langle C|D\rangle &= \text{intersection number, with multiplicities;} \\ \langle\cdot|\cdot\rangle &= \text{bilinear form on divisors.} \end{aligned}$$

- **Néron-Severi group.–** Numerical classes of divisors.

$$\mathrm{NS}(X;\mathbf{Z}) = H^2(X;\mathbf{Z}) \cap H^{1,1}(X,\mathbf{R}).$$

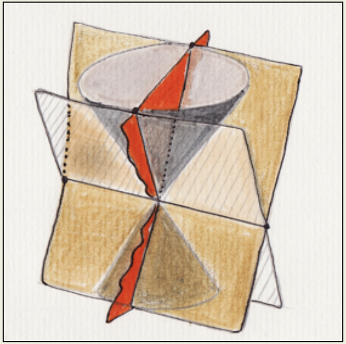
- **Picard number.–** $\rho(X) = \dim_{\mathbf{R}} \mathrm{NS}(X,\mathbf{R})$.

- **Hodge index Theorem.–** *On* $\mathrm{NS}(X;\mathbf{R})$, *the intersection form is non-degenerate, of signature* $(1\,,\ \rho(X)-1)$.



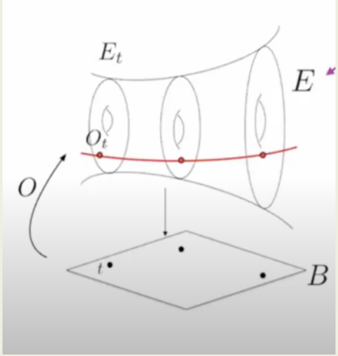
elliptic
f^* has finite order,

parabolic
is virtually unipotent,



loxodromic
or $\lambda(f) > 1$.

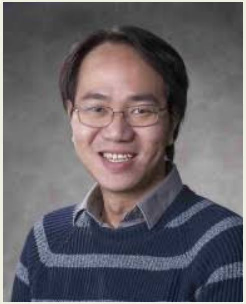
- If f elliptic, then some positive iterate f^k is in $\mathrm{Aut}(X)^0$.



- **Gizatullin's Theorem.–**
If f^* is parabolic, then f preserves a genus 1 fibration $\pi\colon X \to B$, and induces a finite order automorphism of B if X is not an abelian surface.

Examples.– Mordell-Weil groups of a genus 1 fibration $\pi\colon X \to B$: translations from one section of π to another one.

— Break for Questions —



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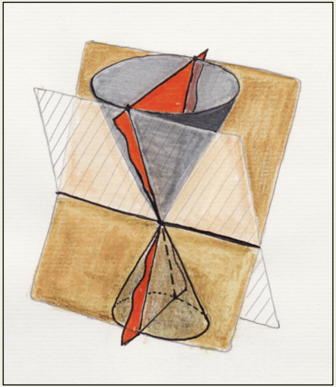
Thank You !

— III —

The invariant measure μ_f:

stable manifolds, periodic points, equidistribution



- Two invariant isotropic lines

$\mathbf{R}\theta_f^+$ and $\mathbf{R}\theta_f^-$, with $\langle\theta_f^+|\theta_f^-\rangle = 1$.

- $f^*\theta_f^+ = \lambda(f)\theta_f^+$.

- $\theta_f^+, \theta_f^- \in \overline{\text{Ample cone}}$.

- θ_f^+ is represented by a closed positive current T_f^+ with $f^* T_f^+ = \lambda(f) T_f^\pm$.

• **Fact.–** The current $T_f^\pm$ is **unique** and has **hölder continous** potentials. The measure

$$\mu_f = T_f^+ \wedge T_f^-$$

is an invariant probability measure.

- **Theorem (Bedford, Lyubich, Smille; C.; Dujardin).–**
  The periodic points of f of period N become equidistributed with respect to μ_f as N goes to $+\infty$:

$$\frac{1}{|Per_f(N)|} \sum_{x \in Per_f(N)} \delta_x \longrightarrow \mu_f .$$

  Moreover, $|Per_f(N)| \simeq \lambda(f)^N$.

- **Theorem (C., Dupont; see also Filip and Tosatti).–**
  If the measure μ_f is smooth, or absolutely continous with respect to the Lebesgue measure on X, then (X, f) is a Kummer example.

- **Kummer groups.–** $\Gamma \subset \mathrm{Aut}(X)$ is a Kummer group if there exists
  - an abelian surface A; a subgroup $\Gamma_A \subset \mathrm{Aut}(A)$;
  - a finite, normal subgroup G of Γ_A;
  - a birational morphism $q_X\colon X \to A/G$;
  - homomorphisms $\tau_X\colon \Gamma \to \mathrm{Aut}(A/G)$ and $\tau_A\colon \Gamma_A \to \mathrm{Aut}(A/G)$;

such that q_X and the quotient map $q_A\colon A \to A/G$ are naturally equivariant and define the same groups:

- $q_X \circ f = \tau(f) \circ q_X$ for every $f \in \Gamma$;
- $q_A \circ g = \tau(f) \circ q_A$ for every $g \in \Gamma_A$;
- $\tau_A(\Gamma_A) = \tau_X(\Gamma_X)$.

— IV —

Periodic orbits for large groups

- **Theorem A (C., Dujardin).–**
  - $\mathbf{k}$ = *number field.*
  - X = *smooth projective surface defined over* $\mathbf{k}$.
  - Γ = *subgroup of* $\mathrm{Aut}(X_{\mathbf{k}})$ *containing parabolic elements with distinct invariant fibrations.*

  If Γ *has a Zariski dense set of periodic points, then* (X, Γ) *is a Kummer group.*

- **Remarks.–**
  - Works also over the field $\mathbf{C}$ if we assume that Γ has no periodic curve.
  - Related question: classify pairs of loxodromic elements with $\mu_f = \mu_g$.
    (see the work of Dujardin and Favre for Hénon automorphisms)

- $\mathbf{k}$ = number field, $\overline{\mathbf{k}} \simeq \overline{\mathbf{Q}}$.
- X and Γ defined over $\mathbf{k}$.
- $\mathrm{Pic}(X;\mathbf{R}) = \mathrm{Pic}(X_{\overline{\mathbf{k}}}) \otimes_{\mathbf{Z}} \mathbf{R}$ (Picard group)

  $=$ Néron-Severi group $\mathrm{NS}(X;\mathbf{R})$ if $\mathrm{Pic}^0(X_{\overline{\mathbf{k}}}) \neq 0$.

- **Definition (A. Baragar).–** A **canonical vector height** is a function

$$h \colon \mathrm{Pic}(X;\mathbf{R}) \times X(\overline{\mathbf{k}}) \to \mathbf{R}$$

such that

(a) for $D \in \mathrm{Pic}(X;\mathbf{R})$, $h(D,\cdot)$ is a Weil height w.r.t. D on $X(\overline{\mathbf{k}})$;
(b) $h(D,x)$ is linear in D: $h(aD+bE,\cdot) = ah(D,\cdot) + bh(E,\cdot)$;
(c) h is equivariant: $h(f^*D,x) = h(D,f(x))$ for all $f \in \Gamma$.

- **Example.–** The Néron-Tate height, for automorphisms fixing the neutral element.
- **Example.–** When $\rho(X) = 2$, and Γ is generated by a loxodromic element (Baragar, after a construction of Silverman).
- **Example.–** Kawaguchi found examples of Wehler surfaces with **no** such height functions.

- **Theorem B (C., Dujardin).–** $\Gamma \subset \mathrm{Aut}(X_{\mathbf{k}})$ *as in Theorem A.*

If there exists a canonical vector height for Γ, then

- *X is an abelian surface,*
- *Γ has a periodic point y,*
- *and h is derived from the Néron-Tate height:*

$$h(D, x+y) = h_{NT}(D, x) + \langle [E] | [D] \rangle \varphi(x).$$

— V —

Proof Strategy

- **1.A– Kawaguchi's stationary height**
  - $\nu =$ probability measure on Γ, with finite support
  - $\sum_f \nu(f) f^*(D) = \alpha(\nu) D$, for some $\alpha(\nu) > 1$, and some D ample

  Then there is a Weil height $\hat{h}_D \colon X(\overline{\mathbf{k}}) \to \mathbf{R}_+$,

  $$\sum_f \nu(f) \hat{h}_D(f(x)) = \alpha(\nu) \hat{h}_D(x), \quad \forall x \in X(\overline{\mathbf{k}}),$$

  with a decomposition as a sum of continuous local heights.
  Finite orbits correspond to points of height 0 for $\hat{h}_D$.

- **1.B– Yuan's equidistribution theorem**, for a sequence of periodic points x_i:

$$\frac{1}{|\Gamma(x_i)|} \sum_{y \in \Gamma(x_i)} \frac{1}{|\mathrm{Gal}(\overline{\mathbf{k}} : \mathbf{k})(y)|} \sum_\sigma \delta_{\sigma(y)} \longrightarrow \mu$$

where μ **is a Γ-invariant probability measure**.

- **2.–** The limit μ does not depend on ν

$$\nu_n \to \frac{1}{2}\delta_f + \frac{1}{2}\delta_{f^{-1}}$$

The measure μ coincides with μ_f, for every loxodromic $f \in \Gamma$.

- **3.–** Compose parabolic elements with distinct invariant fibrations

The measure μ has full support.

- **4.– The measure μ is smooth**

- **5.–** Every loxodromic element is a Kummer example.
  Then (X, Γ) **is a Kummer group**.

What more ?