Arithmetic Holonomicity and Algebraic Dynamics

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A motivation from Algebraic Dynamics

Consider the basic height growth setting: $f : \mathbb{A}^d_K \to \mathbb{A}^d_K$ an endomorphism of an affine space over a global field K, $\lambda : \mathbb{A}^d_K \to \mathbb{A}^1_K$ a regular function (polynomial).

What are the possible growth behaviors for $\left\{h_K(\lambda(f^n(P)))\right\}_{n\in\mathbb{N}}$ *and* $\deg\left(\lambda \circ f^n\right)$*?*

(The height of a forward orbit $P, f(P), f(f(P)), \ldots$ $(P \in \mathbb{A}^d_K(K))$ sampled by λ.)

For the orbits of set-theoretic mappings $f : \mathbb{Z}^d \to \mathbb{Z}^d$, manifestly (pigeonhole) they are either eventually periodic, or else — just by counting the points of a bounded height $h_{\mathrm{aff}}(n_1, \ldots, n_d) := \max_{i=1}^d \log|n_i|$ — it must be that $\left(2\max_{i=0}^n \exp(h_{\mathrm{aff}}(P)) + 1\right)^d > n$. Hence a trivial dichotomy:

> $\limsup_{n\in\mathbb{N}} \frac{h_{\mathrm{aff}}(P)}{\log n} \geq 1/d$ *unless the orbit* $(f^n(P))_{n\in\mathbb{N}}$ *is finite.*

Entering the arithmetic criteria

When the mapping $f : \mathbb{Z}^d \to \mathbb{Z}^d$ is a polynomial, we may reduce the count modulo primes p and observe that $(f^n(P \mod p))_{n\in\mathbb{N}}$ is eventually periodic with both preperiod and period bounded by $\#\mathbb{A}^d(\mathbb{F}_p) = p^d$. Sampling at a polynomial $\lambda : \mathbb{Z}^d \to \mathbb{Z}$, this entails on the generating function $F(x) := \sum_{n\in\mathbb{N}} \lambda(f^n(P))x^n \in \mathbb{Z}[[x]]$ the various mod p rationality conditions: $F \mod p \in \mathbb{F}_p[[x]]$ is rational of degree $\leq p^d$.

Theorem. (Zannier 1981; D. 2013, `ArXiv:1311.4133`) *If $F(x) = \sum_{n\in\mathbb{N}} a_n x^n \in K[[x]]$ has a rational $k(v)$ reduction of degree $d_v(F) \in \mathbb{N}_0 \cup \{\infty\}$ for every non-Archimedean place v of K (write $d_v(f) = \infty$ if F cannot be reduced mod v), then $F(x) \in K(x) \cap K[[x]]$ is rational as soon as*

$$\liminf_{n\in\mathbb{N}} \frac{1}{n} \sum_{v:\, d_v(F)<n/(2+1/\kappa)} \log|k(v)| > \Big(1+\kappa\Big) \limsup_{n\in\mathbb{N}} \frac{1}{n} h_K([a_0 : \cdots a_n])$$

for some $\kappa > 0$.

The arithmetic rationality criterion with mod p conditions

The proof of the criterion is a typical drill in Diophantine approximation: with Siegel's lemma — these are $M = L$ equations in $N = L + \lfloor L/\kappa \rfloor$ variables, bringing in a Dirichlet exponent of $M/(N - M) \sim \kappa$ — one constructs an approximate relation $Q(x)F(x) - P(x) \equiv 0 \mod x^{2L+\lfloor L/\kappa \rfloor}$ with $P, Q \in O_{K,S}[[x]]$ of $\deg P, \deg Q < L + \lfloor L/\kappa \rfloor$ and height estimate $h_K(Q) \leq \kappa \cdot h_K(a_{L+\lfloor L/\kappa \rfloor} : \cdots : a_{L+2L+\lfloor L/\kappa \rfloor}) + O(\log L)$. This is then extrapolated to the full relation $Q(x)f(x) - P(x) = 0$ by the mod v rationality relations (using the trivial "Bézout bound in $\mathbb{P}^d_{k(v)}$") and the product formula in K.

Zannier's paper is: A note on recurrent mod p sequences, *Acta Arithmetica, XLI (1982), pp. 277–280.*

A generalization of Ruzsa's conjecture on pseudo-polynomials: the numerical criterion for rationality should have the best-possible shape

$$\liminf_{n \in \mathbb{N}} \frac{1}{n} \sum_{v \colon d_v(F) < n} \log |k(v)| > \limsup_{n \in \mathbb{N}} \frac{1}{n} h_K([a_0 : \cdots a_n])$$

The application to Algebraic Dynamics on affine space

In our setting we had $d_v(F) \leq \#\mathbb{A}^d(k(v)) = |k(v)|^d$ for all but finitely many $v \in M_K$. Moreover, $h_K([\lambda(P) : \lambda(f(P)) : \cdots : \lambda(f^n(P))]) \leq |S| \sup_{k \leq n} h_{K,\mathrm{aff}}(\lambda(f^k(P))$ since the whole orbit is S-integral. As (PNT)

$$\liminf_{n \in \mathbb{N}} \frac{1}{n} \sum_{v\colon |k(v)| < (n/3)^{1/d}} \log |k(v)| \sim (n/3)^{1/d},$$

the rationality criterion cuts out the dichotomy:

> *Either $\lambda(f^n(P))_{n \in \mathbb{N}}$ is piecewise (on arithmetic progressions) a polynomial in n, or else it has the exponential improvement (with refinement to the individual coordinate λ) over the trivial Northcott bound:*

$$\limsup_{n \in \mathbb{N}} \frac{\log h_{K,\mathrm{aff}}(\lambda(f^n(P)))}{\log n} \geq 1/d.$$

The application to Algebraic Dynamics on affine space

Either $\lambda(f^n(P))_{n\in\mathbb{N}}$ *is piecewise (on arithmetic progressions) a polynomial in* n*, or else it has the exponential improvement (with refinement to the individual coordinate* λ*) over the trivial Northcott bound:*

$$\limsup_{n\in\mathbb{N}} \frac{\log h_{K,\mathrm{aff}}(\lambda(f^n(P)))}{\log n} \geq 1/d.$$

A specialization argument then also leads to an arithmetic proof of this purely algebraic dichotomy on degree growth along a coordinate:

For morphisms $f : \mathbb{A}^d_k \to \mathbb{A}^d_k$ *and* $\lambda : \mathbb{A}^d_k \to \mathbb{A}^1_K$ *over any field* k*, the sequence of degrees* $(\deg \lambda \circ f^{\circ n})_{n\in\mathbb{N}}$ *is either eventually periodic or unbounded.*

References to recent work on this topic

On the algebraic side, Theorem 1.4 in
Urech C.: Remarks on the degree growth of birational transformations, *Math. Res. Lett.*, vol. . **25**, no. 1 (2016).
contains a proof of the full degree bound $\limsup_{n\in\mathbb{N}} \deg f^n \gg n^{1/d}$ (if the sequence is unbounded), *again in this particular setting of endomorphisms* $f : \mathbb{A}_k^d \to \mathbb{A}_k^d$ *of affine space (or more generally, of an endomorphism of any affine variety)*. His method can be seen as an algebraic version of the basic counting (Northcott) lower bound on the full height.

In the same setting, the work
S. Cantat and J. Xie: On degrees of birational mappings, *Math. Res. Lett.*, vol. . **27**, no. 2 (2018).
contains a p-adic proof that also $\liminf_{n\in\mathbb{N}} \deg f^{\circ n} \to \infty$ unless the sequence $(\deg f^{\circ n})_{n\in\mathbb{N}}$ is eventually periodic.

Some backgrond

Consider more generally $f : X \dashrightarrow X$ a dominant rational self-map of any quasi-projective variety X/K (fix also a projective embedding), $P \in X(K)$ a point with well-defined forward f-orbit, and $\lambda : X \dashrightarrow \mathbb{A}^1$ a non-constant rational function. The trivial observation with the Northcott count generalizes to the same lower bound on the *full* height:

$$\limsup_{n\in\mathbb{N}} \frac{h_K(f^{\circ n}(P))}{\log n} \geq \frac{1}{\dim X}.$$

But the improvement to the λ-observable (which surely ought to be true unless the sequence $\lambda(f^n P)$ is finite) has not yet been fully attained. The preceding outline can be applied to yield it in the severely restricted situation that $f : X \to X$ and $\lambda : X \to \mathbb{A}^1_K$ are *morphisms*. In the general case, a *positive* lower bound

$$\limsup_{n\in\mathbb{N}} \frac{h_K(\lambda(f^{\circ n}(P)))}{\log n} > 0$$

was recently attained.

References to recent work

If the sequence $\lambda(f^n P)$ does not become eventually periodic, the lower bound

$$\limsup_{n\in\mathbb{N}} \frac{h_K(\lambda(f^{\circ n}(P)))}{\log n} > 0$$

is proved in:

J. Bell, D. Ghioca and M. Satriano: Dynamical uniform bounds for fibers and a uniform gap conjecture, IMRN (2019),

if $f : X \to X$ is an endomorphism.

(by a p-adic analytic method as opposed to the above " mod p with varying p"), and still more recently and by a completely different method in

J. Bell, L. F. Hu and M. Satriano: Height gap conjectures, D-finiteness and weak dynamical Mordell-Lang Math. Ann. (2020),

for arbitrary $f : X \dashrightarrow X$.

Preparation: the invariants h and τ

Let us introduce the *height* of the formal power series $f(x) = \sum a_n x^n \in K[[x]]$ to be the lim sup featuring above:

$$h_K(f) := \limsup_{n\in\mathbb{N}} \frac{1}{n} h_K([a_0 : \cdots : a_n]).$$

We further introduce a related quantity measuring "almost S-integrality"; it is standard in the theory of G-functions and arithmetic differential equations:

$$\tau_K(f) := \inf_{V\subset M_K,\,\#V<\infty} \limsup_{n\in\mathbb{N}} \frac{1}{n} h_K^{(V)}([a_0 : \cdots : a_n]),$$

where $h_K^{(V)}$ signifies the *V-depletion of the height*: the contributions from V get omitted (and we take the infimum of these depleted heights over an increasing *finite* set of depletion primes). If $K = \mathbb{Q}$, we omit the index and write $h(f)$ and $\tau(f)$.

p-adic analytic information and an algebraic dependency criterion of André

The following is simultaneously the sharp refinement of the arithmetic rationality theorem VIII 1.6 Cor. in André's book *G-Functions and Geometry*, on the one hand, and of the classical Borel-Dwork rationality criterion on the other hand (which is essentially the case $\tau(f) = 0$):

Theorem. *Let $f(x) \in K[[x]]$ have a v-adic radius of meromorphy $M_v(f) > 0$: that means that at each place v one can represent $f(x) = p_v(x)/q_v(x)$ as the quotient of two power series $p_v, q_v \in \mathbb{C}_v[[x]]$ convergent on the disk $|x|_v < R_v$.*
Then $f(x) \in K(x) \cap K[[x]]$ is rational as soon as $\sum_{v \in M_K} \log M_v(f) > \tau_K(f)$.
Reversely, for any collection $(M_v)_{v \in M_K}$, there are continuum-many formal power series $f \in K[[x]]$ convergent on $|x|_v < M_v$ and with $\sum_{v \in M_K} \log M_v = \tau_K(f)$.

In the remainder of the talk, all results that follow are joint work with Frank Calegari and Yunqing Tang

The filtered invariants τ_r and τ_∞. Some examples.

For using the results we discuss below it is also useful to define:

$$\tau_{K,r}(f) := \inf_{V:\,\#V<\infty} \limsup_{n\in\mathbb{N}} \frac{1}{n} h^{(V)}_{K,\mathrm{aff}}(a_{\lfloor(1-1/r)n\rfloor},\ldots,a_n) \le \tau_K(f)$$

and

$$\tau_{K,\infty}(f) := \lim_{r\to\infty} \tau_{K,r}(f).$$

Examples:

- $\tau(\mathrm{Li}_k(x)) = k$, but $\tau_r(\mathrm{Li}_k(x)) = \frac{k}{r}\left(1+\sum_{j=1}^{r-1} 1/j\right)$ — this is k times the asymptotic density of those primes $p \le N$ that divide a number in the range $(1-1/r)N \le n \le N$; — and $\tau_\infty(\mathrm{Li}_k(x)) = 0$.
- In contrast, $\tau_r(\mathrm{Li}_k(x/(x-1))) = \tau_\infty(\mathrm{Li}_k(/(x-1)) = k$ for all k and r;
- $\tau\big((\mathrm{Li}_k(x)^m\big) = k\sum_{j=1}^{m} 1/j$ for all k and m, and the same holds for the τ_∞ iff $m \ge 2$.

The Template-Denominator problem

We follow André in a more general setting of simultaneous meromorphic uniformization of $x(z)$ and $f(x(z))$ by disks in the v-adic z-plane (but the rationality criterion above will only hold for the actual meromorphic radii M_v; as the example of $\log(1-x)$ already shows). Furthermore it is critical for the proofs to work within the higher-dimensional setting.

Notation: $\mathbb{C}_v$: the completion of an algebraic closure of K_v.

$$D_d(\mathbf{0}, R_v) := \{\mathbf{z} = (z_1, \ldots, z_d) \in \mathbb{C}_v^d \ : \ \max_{1 \leq i \leq d} |z_i|_v < R_v\}.$$

Extend the definition of h, τ, τ_r and τ_∞ to the multivariate setting *by using the total degree*, in the obvious way: e.g.,

$$h_K(f) := \limsup_{n \in \mathbb{N}} \frac{1}{n} h_K((a_{\mathbf{n}})_{\mathbf{n}:|\mathbf{n}| \leq n})$$

The Template-Denominator problem

Thus, for all places $v \in M_K$ we consider:

- a radius $R_v > 0$;
- a *holomorphic* (convergent power series) mapping $\mathbf{x}_v(\mathbf{z}) : D_d(\mathbf{0}, R_v) \to \mathbb{C}_v^d$, normalized by $\mathbf{x}_v(z) = \mathbf{z} + O(\mathbf{z}^2)$, and such that $\mathbf{x}_v(\mathbf{z}) = \mathbf{z}$ for all but finitely many v.

We will call this datum here an *adelic template*. Its *solutions* are those formal power series $f(\mathbf{x}) \in K[[\mathbf{x}]]$ of the property that, for every $v \in M_K$, the formal power series

$$f(\mathbf{x}_v(\mathbf{z})) \in \mathbb{C}_v[[\mathbf{z}]]$$

is the germ of a v-adic meromorphic function on the polydisk $D_d(\mathbf{0}, R_v)$.

Our basic problem — as it turns out, of a Diophantine nature — will be to find the all the solutions $f(\mathbf{x})$ to a given template subject to various denominator constraints on the $\tau(f)$ or $\tau_r(f)$ or the more precise denominator type of f.

The rationality theorem — a restatement — and its extension to holonomicity

We can restate more tersely in our terminology. We will see below a more precise quantitative version of this basic holonomicity theorem that is critical for the applications.

Theorem on Rationality. *For the trivial adelic template* $x_v(z) = z$ *with radii* $(R_v)_{v \in M_K}$*, the formal solutions* $f(x) \in K[[x]]$ *of* $\tau_K(f) < \sum_{v \in M_K} \log R_v$ *are* **exactly** *the rational power series* $f(x) \in K(x) \cap K[[x]]$.

Theorem on Holonomicity. *For any adelic template* $\big(x_v(z) = z + \cdots\big)_{v \in M_K}$ *with radii* $(R_v)_{v \in M_K}$*, all formal solutions* $f(x) \in K[[x]]$ *with* $\tau_{K,\infty}(f) < \sum_{v \in M_K} \log R_v$ *are necessarily D-finite (alias holonomic: formal solutions of a linear ODE over* $K(x)$*).*

The Diophantine method of André

The following is essentially André's Main Criterium VIII 1.6 in *G-Functions and Geometry*, with a mild refinement to the invariant τ_r.
(NB: The set $V \subset M_K$ in *loc. cit.* must be assumed to have $\#V < \infty$.)

André's criterion. *Let $m \in \mathbb{N}$, and suppose that $\mathbf{x}_v(\mathbf{z}) : D_d(\mathbf{0}, R_v) \to D_d(\mathbf{0}, S_v)$ (thus, $S_v \geq R_v$ by Schwarz's lemma, with equality in the case of a trivial template). If there is a real parameter $\kappa \in \mathbb{R}^{>0}$ such that the inequality*

$$\sum_{v \in M_K} \log R_v > \tau_{K,m}(f_1, \ldots, f_m) + \kappa \cdot h_K(f_1, \ldots, f_m) + \Big(\frac{1}{m}\big(1 + 1/\kappa\big)\Big)^{1/d} \cdot \sum_{v \in M_K} \log^+ S_v$$

is fulfilled, then the adelic template $\big(\mathbf{x}_v(\mathbf{z})\big)_{v \in M_K}$ has at most $m-1$ solutions linearly independent over $K(\mathbf{x})$.

The parameters design

Condition in the general André theorem:

$$\sum_{v \in M_K} \log R_v > \tau_K + \kappa \cdot h_K + \left(\frac{1}{m} (1 + 1/\kappa) \right)^{1/d} \cdot \sum_{v \in M_K} \log^+ S_v$$

Our basic goal will be to delete the terms involving the parameter κ. (Optimization in κ is not good enough for neither the rationality theorem nor for applications to transcendence.) Before we take it up in a moment, some intuitive guidelines:

- $\sum_{v \in M_K} \log R_v > \tau_K$ is the essential positivity condition;
- the proof as before is by an auxiliary (Siegel lemma) linear form $Q_1 f_1 + \cdots + Q_m f_m$ vanishing highly at the origin, with polynomials Q_i of heights bounded proportionally to κh_K and degrees bounded proportionally to $\left(\frac{1}{m} (1 + 1/\kappa) \right)^{1/d}$.
- the $\log^+ S_v$ term arises from estimating $\sup_{|\mathbf{z}|_v = R_v} |Q_i(\mathbf{x}_v(\mathbf{z}))|$.

Proof of André's criterion: the Siegel lemma part

Fix parameters $\alpha \in \mathbb{N}$ and $\kappa \in \mathbb{R}^{>0}$. Asymptotically in α for the fixed $(f_1, \ldots, f_m)$, there exists an m-tuple of polynomials $Q_1, \ldots, Q_m \in K[\mathbf{x}]$, not all zero, such that:

(i) $\max_{i=1}^m \deg Q_i \leq \left(\frac{1}{m}\left(1 + \frac{1}{\kappa}\right)\right)^{\frac{1}{d}} \alpha + o(\alpha)$;

(ii) $\max_{i=1}^m h_K(Q_i) \leq \kappa \cdot h_K(f_1, \ldots, f_m) \cdot \alpha + o(\alpha)$;

(iii) $Q_1 f_1 + \cdots + Q_m f_m \in (x_1, \ldots, x_d)^\alpha$ *(order of vanishing $\geq \alpha$ at the origin).*

We have $\binom{\alpha+d-1}{d} \sim \alpha^d/d!$ linear equations in the $m\binom{N+d}{d} \sim mN^d/d!$ unknown coefficients. The degree parameter choice

$$N \sim \left(\frac{1}{m}\left(1 + \frac{1}{\kappa}\right)\right)^{\frac{1}{d}} \alpha > m^{-1/d}\alpha$$

insures that the solution space is non-zero, and brings in a Dirichlet exponent of $\sim \kappa\alpha$ in Siegel's lemma.

Proof of André's criterion: the extrapolation

We next push this Siegel lemma approximation to a full $K(\mathbf{x})$-linear dependency

$$U := Q_1 f_1 + \cdots + Q_m f_m = 0.$$

If this power series $U(\mathbf{x})$ is not zero, let $\beta \geq \alpha$ be the minimum degree of a non-vanishing term, and choose a multi-index $\mathbf{k} \in \mathbb{N}_0^d$ with

$$\eta := \frac{1}{\mathbf{k}!} \frac{\partial^{\mathbf{k}} U(\mathbf{x})}{\partial \mathbf{x}^{\mathbf{k}}}\Big|_{\mathbf{x}=\mathbf{0}} = \frac{1}{\mathbf{k}!} \frac{\partial^{\mathbf{k}} U(\mathbf{x}_v(\mathbf{z}))}{\partial \mathbf{z}^{\mathbf{k}}}\Big|_{\mathbf{z}=\mathbf{0}} \neq 0.$$

We choose a large enough finite subset $V \subset M_K$ of the places, and estimate the prime-to-V part of η trivially by the Liouville estimate:

$$\sum_{v \notin V} \log |\eta|_v \leq \max_{1 \leq i \leq m} h_K^{(V)}(Q_i) + \beta \cdot h_K^{(V)}(f_1, \ldots, f_m) + o(\beta). \quad (1)$$

Proof of André's criterion: the extrapolation

The second term in this estimate (1) is proportional-asymptotic to the inevitable "denominators part" $\tau_K(f_1, \ldots, f_m)$.
At the finitely many places $v \in V$, we use a stronger estimate coming by way of the v-adic analytic representation

$$q(\mathbf{z})U(\mathbf{x}_v(\mathbf{z})) = Q_1(\mathbf{x}_v(\mathbf{z}))h_1(\mathbf{z}) + \cdots + Q_m(\mathbf{x}_v(\mathbf{z}))h_m(\mathbf{z}) \in \mathbb{C}_v[[\mathbf{z}]],$$

$$q(\mathbf{0}) = 1; \quad \text{thus still } \eta = \frac{1}{\mathbf{k}!}\frac{\partial^{\mathbf{k}}(q(\mathbf{z})U(\mathbf{x}_v(\mathbf{z})))}{\partial \mathbf{z}^{\mathbf{k}}}\Big|_{\mathbf{z}=\mathbf{0}},$$

where now $\mathbf{x}_v(\mathbf{z}), q(\mathbf{z})$ and $h_i(\mathbf{z})$ are holomorphic on the v-adic polydisk $\|\mathbf{z}\|_v := \max_{i=1}^{d} |\mathbf{z}|_v \leq R_v$. (Shrink the R_v a little bit if necessary.)

Proof of André's criterion: the extrapolation

On the boundary torus $\|\mathbf{z}\|_v = R_v$ we have the estimate

$$\log|q(\mathbf{z})U(\mathbf{x}_v(\mathbf{z}))|_v \le \log\left(m\binom{N+d}{d}\right) + N\log^+ S_v$$
$$+ \max_{1\le i\le m}\sup_{\|\mathbf{z}\|_v=R_v}\log^+|h_i(\mathbf{z})|_v + \sup_{\|\mathbf{z}\|_v=R_v}\log^+|q(\mathbf{z})|_v + \max_{1\le i\le m} h_v(Q_i).$$

Here

$$N := \max_{1\le i\le m}\deg Q_i \le \left(\frac{1}{m}\left(1+\frac{1}{\kappa}\right)\right)^{\frac{1}{d}}\beta + o(\beta)$$

in Siegel's lemma. We use this to estimate

$$\eta = \frac{1}{(2\pi i)^d}\int_{\|\mathbf{z}\|=R_v} q(\mathbf{z})U(\mathbf{x}_v(\mathbf{z}))\frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{k}}}$$

[this Cauchy integral formula is for the Archimedean case; analogous appeal to the maximum principle in the ultrametric case]

Proof of André's criterion: the extrapolation

At this point we use Schwartz's lemma: as the integrand $q(\mathbf{z})U(\mathbf{x}(\mathbf{z}))/\mathbf{z}^{\mathbf{k}}$ is holomorphic, and $|\mathbf{k}| = \beta$, the Cauchy estimate yields

$$\log|\eta|_v \leq \left(\frac{1}{m}\left(1+\frac{1}{\kappa}\right)\right)^{\frac{1}{d}}\beta + \max_{1\leq i\leq m} h_v(Q_i) - \beta\log R_v + o(\beta). \quad (2)$$

Now if $V \subset M_K$ and $\alpha \leq \beta$ are large enough, this contradicts André's inequality upon adding (1) (on $M_K \setminus V$) to (2) over all $v \in V$. Thus André's condition forces identical vanishing $Q_1 f_1 + \cdots + Q_m f_m \equiv 0$, completing the proof of the André's criterion, and of the holonomicity theorem. ∎

The proof of the rationality theorem

We add a multivariate amplification to André's criterion, that only works in the case of the trivial template $x_v(z) = z$. Introduce d auxiliary variables $\mathbf{x} := (x_1, \ldots, x_d)$ and take:

$$f_I(\mathbf{x}) := \prod_{(i,j)\in I} f(x_i x_j) \in K[[x]]$$

for any subset $I \subset \{1, \ldots, d\}$ of pairwise disjoint pairs (i, j) of indices. Clearly $m := \#\{\text{available } I\} > 2^{(d/2)^2}$: the key is that the number of power series thus constructed out of the starting $f(x)$ is super-exponential in the number d of auxiliary variables.

The proof of the rationality theorem

Whereas $\tau(f(x)^n)$ can be as large as $\tau(f(x)) + \sum_{j=1}^{n-1} 1/j$ — unless $\tau(f) = 0$, that was essentially the S-integrality condition in the original Borel-Dwork criterion, — for *disjoint* variables we always have $\tau(f(x_1)\cdots f(x_n)) = \tau(f)$. Additionally, all $f(x_i x_j)$ — and the products we have attached thereof — of the trivial adelic template of radii $R_v^{1/2}$ have a correspondingly decreased height by the same factor of $1/2$:

$$h(f(xy)) = h(f(X))/2, \quad \tau(f(xy)) = \tau(f(X))/2.$$

We have $m > 2^{d^2/4}$ in:

$$\sum_{v \in M_K} \log M_v > \tau_K(f) + \kappa \cdot h_K(f) + \left(\frac{1}{m}\big(1 + 1/\kappa\big)\right)^{1/d} \cdot \sum_{v \in M_K} \log^+ M_v$$

So we get the rationality of $f(x)$ by firstly taking $\kappa \to 0$ (that kills the second term in the RHS), and then $d \to \infty$ (that kills the third term in the RHS because $m^{-1/d} \to 0$ in this construction.) ■

The arithmetic holonomicity theorem

So the solutions with $\sum_{v\in M_K}\log R_v > \tau_K(f)$ are necessarily holonomic, and the proof returns an effective estimate on the order of the minimal ODE. The crucial points for applications is a tight estimate.

Theorem. (Calegari, Tang, D., 2020) *The solutions with*

$$\sum_{v\in M_K}\log R_v > \tau_K(f)$$

satisfy a linear inhomogeneous ODE "$L(f)\in K[x]$" where the order $r-1$ of the differential operator $L\in K\left[x, x\frac{d}{dx}\right]$ fulfills

$$r \leq \frac{\sum_v \log^+ S_v}{\sum_v \log R_v - \tau_{K,r}(f)}.$$

A variant with denominator types

Consider for simplicity $f(x) = \sum a_n x^n \in \mathbb{Q}[[x]]$ where, for some $s \in \mathbb{R}^{>0}$ and $t \in \mathbb{N}$:

> $(\star_{s,t})$: $[1, \ldots, sn]^t \, a_n \in \mathbb{Z}[1/N_f]$, *with some positive integer* $N = N_f$ *allowed to depend on the formal solution* $f(x)$.

Then $\tau(f) \leq st$ in these conditions, and we have the variant by the compatibility of these denominators:

Theorem. (Calegari, Tang, D., 2020) *The solutions with*

$$\sum_v \log R_v > st$$

of the adelic template $x_v(z) = z + \cdots : D(0, R_v) \to D(0, S_v)$ *of* $\mathbb{Q}$ *with denominator types* $(\star_{s,t})$ *have a finite-dimensional* $\mathbb{Q}(x)$*-linear span of dimension at most*

$$\leq \frac{\sum_v \log^+ S_v}{\sum_v \log R_v - st}$$

over $\mathbb{Q}(x)$*.*

Proof of the holonomicity theorem

We introduce a similar multivariate Diophantine amplification by replacing the original variety $\mathbb{A}^1_K$ by its d-th power $\mathbb{A}^d_K$, and take a $d \to \infty$ asymptotic. But this time, in the case that the adelic template is non-trivial, we may no longer use the $f(x_i x_j)$: the function $f(x(z)x(y))$ is no longer analytic on the disk $|z| < R_v^{1/2}$. So the best we can do is to take the disjoint variable products of the $f(x_i)$ or their derivatives.

We highlight again the crucial piece:

Lemma. *Consider two non-zero formal power series $f(\mathbf{x}) \in K[[\mathbf{x}]] \setminus \{0\}$ and $g(\mathbf{y}) \in K[[\mathbf{y}]] \setminus \{0\}$ in the disjoint blocks of variables $\mathbf{x}$ and $\mathbf{y}$. Then the product series*

$$H(\mathbf{x}, \mathbf{y}) := f(\mathbf{x})g(\mathbf{y}) \in K[[\mathbf{x}, \mathbf{y}]]$$

has

$$h_K(H) = h_K(f, g) \quad and \quad \tau_K(H) = \tau_K(f, g).$$

(Multiplication has no 'carries' when the variables are disjoint.)

Proof of the holonomicity theorem

Suppose for contradiction that the derivatives $f, f', f'', \dots, f^{(r-1)}$ are $K(x)$-linearly independent with some

$$(\star) \qquad \sum_v \log R_v > \tau_{K,r}(f) + \frac{1}{r} \sum_v \log^+ S_v,$$

and apply the André's criterion (version above) to the $m = r^d$ solutions $(0 \leq j_i < r)$

$$\left(\frac{d}{dx_1}\right)^{j_1} f(x_1) \cdots \left(\frac{d}{dx_d}\right)^{j_d} f(x_d) \in K[[x_1, \dots, x_d]].$$

of the d-th Cartesian power adelic template $\mathbf{x}_v(\mathbf{z}) := (x_v(z_1), \dots, x_v(z_d))$. The κ term from Siegel's lemma goes away in the $d \to \infty$ limit, and the resulting inequality is exactly the opposite of $(\star)$. ∎

A comparison: Zudilin's determinantal criterion

In *A determinantal approach to irrationality* (Constr. Approx., 2017), Zudilin gave a determinantal criterion for irrationality proofs. In effect his result is a *rationality* criterion in terms of the generating function $f \in \mathbb{Q}[[x]]$ in the involved linear forms $r_n = a_n - \xi b_n$ where ξ is the constant whose irrationality is to be proved. In our terms, he establishes the rationality of the solutions $f(x) \in \mathbb{Q}[[x]]$ to the following template-denominator problem:

- $f\left(z/(1 - \frac{z}{4r})^2\right) \in \mathbb{C}[[z]]$ is convergent on the complex disk $|z| < 4r$ (that means precisely f is holomorphic on $\mathbb{C} \setminus (-\infty, -r]$;
- $f(x)$ is convergent on $|x|_p < 1$ for every prime p; and
- the domain of analyticity is large with respect to the denominators: $\log(4r) = \sum_v \log R_v > \frac{3}{2}\tau(f)$.

But the example — necessarily holonomic! — of $f(x) = \log(1+x)$, with $r = 1$ and $\tau = 1$, demonstrates that the coefficient $3/2$ may not be dropped to below the value $\log 4$. (*An open problem is: What is the optimal coefficient?*)

A case of full resolution for a Denominator-Singularity problem: an arithmetic characterization of arcsin among all G-functions

Among the classical hypergeometric series, our invariants are easily calculated. E.g., for all primes p, we have $\tau({}_2F_1(\{1,1\},\{1/p\};x)) = p$ with denominator LCM of $[1,\ldots,pn+1]$ for the first n coefficients; a p-adic convergence radius of $p^{p/(p-1)}$; and a unit convergence radius at the other primes. Let us try $p=2$ and observe that

$$ {}_2F_1(\{1,1\},1/p;x) = \frac{1+\sqrt{\frac{x}{x-1}}\arcsin\sqrt{x}}{x-1}. $$

Equivalently: whereas $\arcsin x$ has 2-adic convergence radius only $1/2$, and the two individual factors in

$$ H(x) := \sqrt{1-x}\cdot\sqrt{x}\arcsin\sqrt{x} \in \mathbb{Q}[[x]] $$

have 2-adic radii of just $1/4$, this product $H(x)$ (a solution of an inhmogeneous order-1 ODE) has a 2-adic radius 16 times as large.

A case of full resolution for a Denominator-Singularity problem: arithmetic characterization of arcsin among the G-functions

We observe that if $f(x) \in \mathbb{C}[[x]]$ is a formal solution at the origin of a linear ODE with singularities limited only to $0, 1$ and ∞, then not only is this branch $f(x)$ holomorphic on the slit plane $\mathbb{C} \setminus [1, +\infty)$ (of conformal radius $R_\infty = 4$), but there is an even larger disk $D(0, 4\pi)$ where x and $f(x)$ admit a simultaneous analytic uniformization: the map $x_\infty(z) := 1 - \exp(-iz/(1 + z^2/(16\pi^2))) = z + \cdots$ turns $f(x(z))$ automatically holomorphic (convergent) on that disk. For any strictly smaller radius $R < 4\pi$, there is a positive computable $c(R) > 0$ such that the same holds on $D(0, R)$ for all linear ODE without singularities outside the union of the three disks $|x| < c(R)$, $|1 - x| < c(R)$ and $|1/x| < c(R)$.

A case of full resolution for a Denominator-Singularity problem: an arithmetic characterization of arcsin among all G-functions

Recall the above function $H(x) := \sqrt{1-x} \cdot \sqrt{x} \arcsin \sqrt{x} \in \mathbb{Q}[[x]]$. It has singularities only at $0, 1$ and ∞, a denominator type $[1, \ldots, 2n+1]$ and, along with $H(x/(x-1))$, it it is a member of the finite-dimensional (since $\log 4\pi + \log 4 > 2$) $\mathbb{Q}[x]$-linear space of such holonomic functions.
We are in a position to fully determine all such functions.

An arithmetic characterization of $H(x) = \sqrt{x - x^2}\arcsin\sqrt{x} \in \mathbb{Q}[[x]]$

Theorem

The formal solutions $f(x) = \sum_n a_n x^n \in \mathbb{Q}[[x]]$ to the joint conditions

(i) *$f(x)$ converges on the 2-adic disk $|x|_2 < 4$ as well as on the p-adic disk $|x|_p < 1$ for all p;*

(ii) *$[1, \ldots, 2n+1]a_n \in \mathbb{Z}[1/N_f]$, for some positive integer N_f (allowed to depend on the solution f); and*

(iii) *for some non-zero polynomial $q(x) \in \mathbb{C}[x] \setminus \{0\}$, the formal function $q(x)f(x)$ satisfies a linear ODE $L(f) = 0$ with some linear differential operator L whose singularities are limited to the union of the three disks $|x| < c$, $|1-x| < c$ and $|1/x| < c$ ($c > 0$ a certain small enough absolute constant),*

are **exactly** *the three-dimensional space*

$$\mathbb{Q}[[x]] \cap \mathrm{span}_{\mathbb{Q}(x)}\{1, H(x), H(x/(x-1))\}.$$

Recall the above function $H(x) := \sqrt{1-x} \cdot \sqrt{x} \arcsin \sqrt{x} \in \mathbb{Q}[[x]]$. It has singularities only at $0, 1$ and ∞, a denominator type $[1, \ldots, 2n+1]$ and, along with $H(x/(x-1))$, it is a solution the following choice of adelic template (where the radii are chosen for the convenience in the proof):

- $R_\infty := 8\pi/(2+\pi) = 4.888\ldots$ and $x_\infty(z) := 1 - \exp(-iw/(1+w^2/(16\pi^2)))$ followed by the unit derivative conformal isomorphism $w(z) = iz/(1+(\pi-2)z/(8\pi)) = iz + \cdots$ from $D(0, R_\infty)$ onto the disk of diameter $[-4i, 2\pi i]$;
- All non-Archimedean $x_p(z) = z$, with radii $R_2 = 4$ and $R_p = 1$ for odd p.

An arithmetic characterization of $H(x) = \sqrt{x - x^2} \arcsin \sqrt{x}$: the proof outline

Our template choice has $S_\infty = |x_\infty(-4i)| = 84.71\ldots$ and $S_p = R_p$ for the finite p, hence the quotient

$$\frac{\sum_v \log^+ S_v}{\sum_v \log R_v - 2} < \frac{\log 84.72 + \log 4}{\log 4.887 + \log 4 - 2} = 5.988\ldots < 6.$$

This proves that the solution space is at most 5-dimensional over $\mathbb{Q}(x)$. At this point we can easily finish off the proof by observing that the solutions come in quadruples $U(x), U'(x), U(x/(x-1))$ and $U'(x/(x-1))$, and by analyzing the $\mathbb{Q}(x)$-linear dependencies among these. First off, three $\mathbb{Q}(x)$-linearly independent solutions $U(x), U'(x)$ and $U''(x)$ already contradict this dimension estimate, so we are reduced to the case that $U(x)$ fulfills as second order homogeneous linear ODE. The case of $\{0, 1\infty\}$-singularities then become exactly the hypergeometric equation, which is easily analyzed yielding precisely the 3-dimensional solution space that we found. The remaining order 2 cases are now easily ruled out.

A brief outline of an application to the irrationality of the 2-adic period $\zeta_2(5) \in \mathbb{Q}_p \setminus \mathbb{Q}$

This was our original motivation. It follows Calegari's p-adic avatar of the Apéry-Beukers method for $\zeta(3)$:
Calegari F.: Irrationality of certain p-adic periods for small p, *IMRN, no. 20 (2005), pp. 1235–1249.*
We work on the modular curve $X_0(2) \cong \mathbb{P}^1$ with the Hauptmodul

$$x = x(q) := \frac{\Delta(2\tau)}{\Delta(\tau)} = q \prod_{n=1}^{\infty} (1 + q^n)^{24},$$

in which we may formally expand

$$q = x - 24x^2 + 852x^3 - 35744x^4 + \cdots \in x + x^2\mathbb{Z}[[x]].$$

In this coordinate, it is readily seen that the ordinary component of the cusp ∞ is just the unit disk $|x|_2 \leq 1$. As the Fricke involution w_2 swaps $2^{12}x$ with $1/x$, it follows that the ordinary component of the other cusp 0 is given by $|x|_2 \geq 2^{12}$.

A brief outline of an application to the irrationality of the 2-adic period $\zeta_2(5) \in \mathbb{Q}_p \setminus \mathbb{Q}$

We multiply out the two opposite weight, overconvergent (one classical, the other a "true" 2-adic modular form) 2-adic Eisenstein series $E_4 \in \mathbb{Q} + q\mathbb{Z}[[q]] = \mathbb{Q} + x\mathbb{Z}[[x]]$ and $E_{-4} \in \zeta_2(5)/2 + q\mathbb{Q}[[q]] = \zeta_2(5)/2 + x\mathbb{Q}[[x]]$, where the non-constant piece $E'_{-4} \in \mathbb{Q}[[x]]$ of E_{-4} has $\tau(E_{-4}) = 5$. Individually the two U_p-eigenforms E_4 and E_{-4} are both only convergent on the modest disk $|x|_2 \leq 1$, but by

> *Buzzard K.: Analytic continuation of overconvergent eigenforms, J. Amer. Math. Soc., vol.* **16**, *no. 1, pp. 29–55,*

they are rigid-analytic section of the respective sheaves $\omega^{\otimes 4}$ and $\omega^{\otimes(-4)}$ over the much larger locus $|x|_2 \leq 2^{12}$, and so their product, a section of $\omega^{\otimes 4} \otimes \omega^{\otimes(-4)} \cong \mathcal{O}_{X_0(2)}$, is convergent on $|x|_2 \leq 2^{12}$.

A brief outline of an application to the irrationality of the 2-adic period $\zeta_2(5) \in \mathbb{Q}_p \setminus \mathbb{Q}$

This time we select:

- $R_p := 1$ and $x_p(z) = z$, if $p \notin \{2, \infty\}$;
- $R_2 := 2^{12}$ at the 2-adic place and $x_2(z) = z$;
- $R_\infty := 1/5$ at the Archimedean place and

$$x_\infty(z) := x(q(z)) = x(z/(1+3z)) = \frac{z}{1+3z} \prod_{n=1}^{\infty} \Big(1 + \big(\frac{z}{1+3z}\big)^n\Big)^{24},$$

where the crucial point with the preliminary Möbius transformation

$$q(z) : \{z \in \mathbb{C} : |z| < 1/5\} \to B := \{q \in \mathbb{C} : |q + 3/16| < 5/16\},$$

$$z \mapsto \frac{z}{1+3z} = z + z^2 \sum_{n=0}^{\infty} (-3)^{n+1} z^n,$$

is that it shifts the disk to the left, towards the boundary zero $0 = x(-1)$ and away from the boundary pole $\infty = x(1)$, all the while keeping the adequately large conformal radius of $1/5$.

A brief outline of an application to the irrationality of the 2-adic period $\zeta_2(5) \in \mathbb{Q}_p \setminus \mathbb{Q}$

- $R_p := 1$ and $x_p(z) = z$, if $p \notin \{2, \infty\}$;
- $R_2 := 2^{12}$ at the 2-adic place and $x_2(z) = z$;
- $R_\infty := 1/5$ at the Archimedean place and

$$x_\infty(z) := x(q(z)) = x(z/(1+3z)) = \frac{z}{1+3z} \prod_{n=1}^{\infty} \Big(1+(\frac{z}{1+3z})^n\Big)^{24},$$

At this point the proof is a numerical piece of luck: the arithmetic holonomicity quotient

$$\frac{\sum_v \log^+ S_v}{\sum_v \log R_v - 5} = \frac{12\log 2 + \log^+ |x(1/8)|}{12\log 2 + \log(1/5) - 5} < \frac{9.5}{6.7-5} = 5.58\ldots < 6,$$

but one shows easily that the functions $1, H', H'', H''', H^{(iv)}$ and H^{v} (indeed holonomic of the minimal order 6) are $\mathbb{C}(x)$-linearly independent.

Thank you! Stay safe and take care!

Appendix on normalizations

Although the cases of interest to us occur with $K = \mathbb{Q}$, we work over a global field K. Normalize the absolute values $|\cdot|_v$, $v \in M_K$ in the usual way ($|\alpha|_v$ is the "module" reflecting the change in Haar measure of K_v^+ under $x \mapsto \alpha x$), so that the product formula holds. We extend our notation to $f(\mathbf{x}) = \sum a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in K[[x_1, \ldots, x_d]]$. For $V \subset M_K$, let

$$h_K([\alpha_0 : \ldots : \alpha_N]) := \sum_{v \in M_K} \max_{0 \le i \le N} \log |\alpha_i|_v$$

$$h_K^{(V)}([\alpha_0 : \ldots : \alpha_N]) := \sum_{v \in M_K \setminus V} \max_{0 \le i \le N} \log |\alpha_i|_v$$

$$|\mathbf{n}| := n_1 + \cdots + n_d; \quad h_K(f) := \limsup_{n \in \mathbb{N}} \frac{1}{n} h_K((a_{\mathbf{n}})_{\mathbf{n} : |\mathbf{n}| \le n})$$

$$h_K^{(V)}(f) := \limsup_{n \in \mathbb{N}} \frac{1}{n} h_K^{(V)}((a_{\mathbf{n}})_{\mathbf{n} : |\mathbf{n}| \le n})$$

$$\tau_K(f) := \inf_{V : \# V < \infty} h_K^{(V)}(f).$$