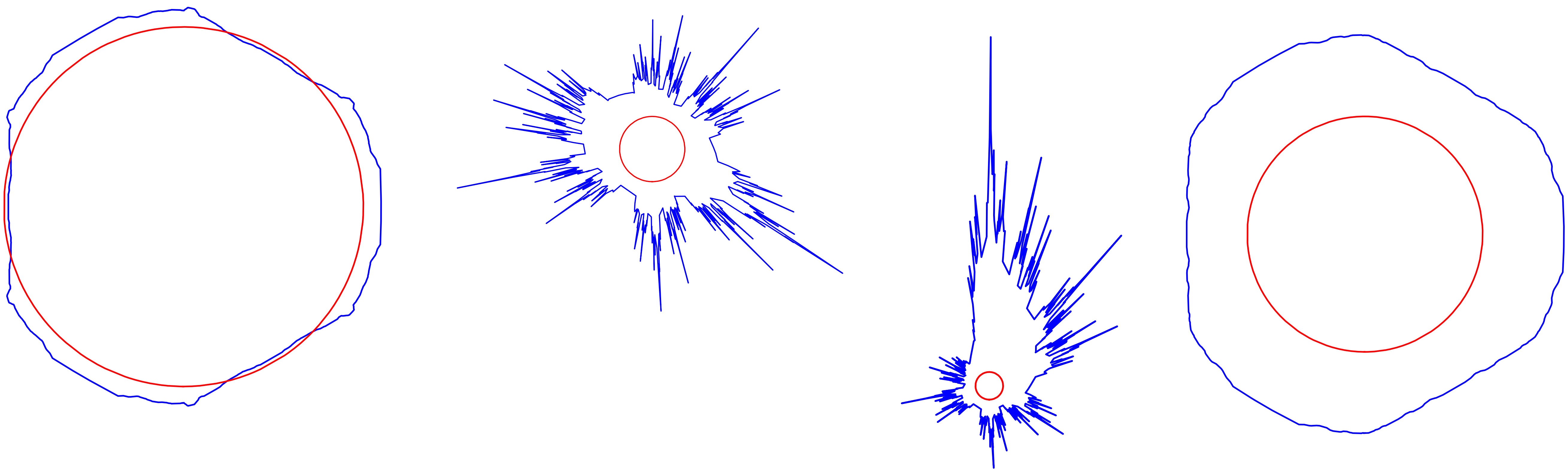
Equivariant currents and heights on the boundary of the ample cone of a K3 surface



Simion Filip, University of Chicago

joint with Valentino Tosatti

Setup

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K3 surfaces

X algebraic surface with nowhere vanishing 2-form Ω

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$\rho = \mathrm{rk} N$

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Standing assumptions (simplifying):

$\rho \geq 3$

$\mathrm{Aut}(X) \to \mathrm{SO}(N) \backsimeq \mathrm{SO}_{1,\rho-1}(\mathbb{R})$ gives a lattice

Singular fibers of elliptic fibrations are reduced and irreducible

Example

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$$X : (1 + x^2)(1 + y^2)(1 + z^2) - 5xyz = 1 \text{ in } \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

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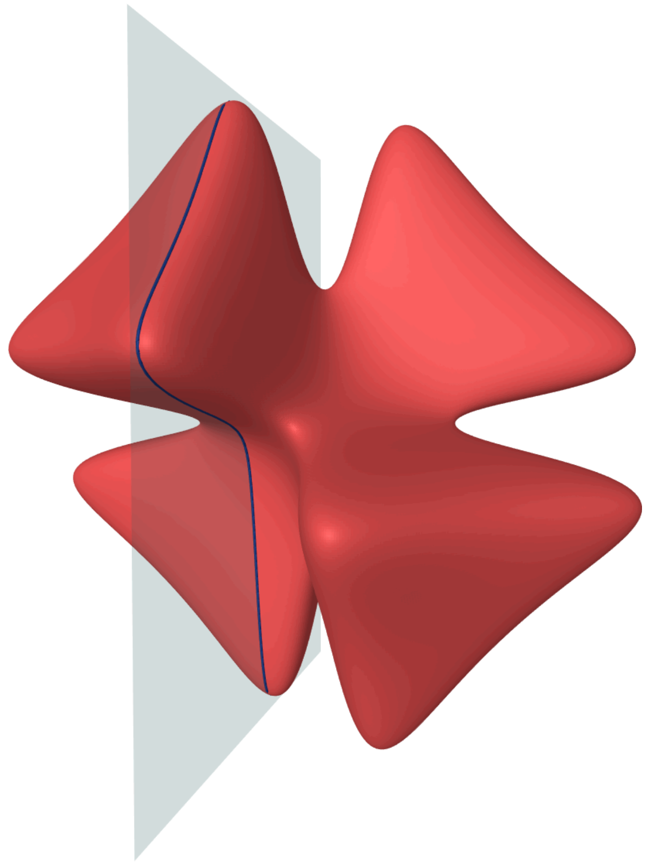

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Ample Cone

$$\partial \operatorname{Amp}(X) \leftarrow \partial^{\circ} \operatorname{Amp}_c(X)$$ (TBE)

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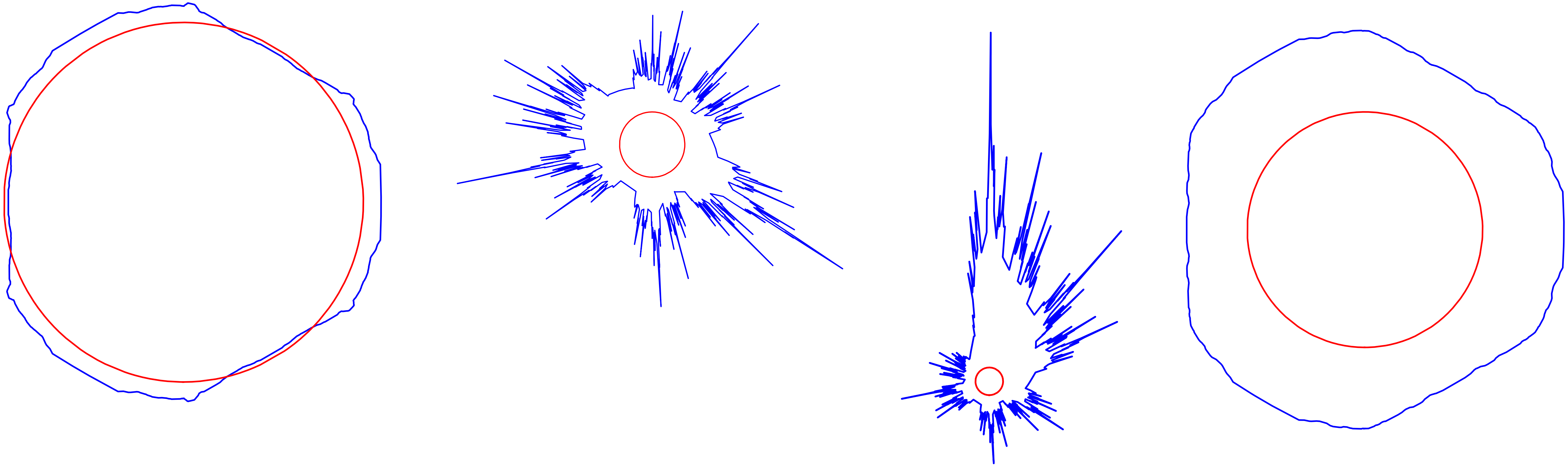
Theorem H $\exists!h^{can} : \partial^\circ \operatorname{Amp}_c(X) \to \mathscr{H}eights(X)$

- equivariant
- agrees with Silverman's canonical height for classes expanded by hyperbolic automorphisms
- $\forall p \in X(\overline{\mathbb{Q}})$ the function $h^{can}_{\alpha}(p)$ is continuous in α



Projection of region where $h_{\alpha}^{can}(p) = 1$

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$$b \mapsto h^{can}_{X_b, \sigma_0(b)}(\sigma_1(b)) \in \mathbb{R}_{\geq 0}$$

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Variant (forget sections):

$$\mathrm{Aut}_\pi(X) \times \mathrm{Pic}^{rel}_\pi(X) \to \mathscr{H}eights(B) \to \mathrm{Pic}(B)$$

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Useful Lemma (F.-Tosatti):

L^0 line bundle of π-relative degree 0 on X.

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Archimedean variant (F.-Tosatti): cf. Betti form

Smooth closed ω on X s.t. $\displaystyle\int_{X_b} \omega = 0$

There exists **continuous** ϕ s.t. $\omega + dd^c\phi\,|_{X_b} \equiv 0$

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What is $\partial^\circ \operatorname{Amp}_c(X)$?

Blow up $\partial \operatorname{Amp}(X)$ at the rational rays.

i.e. add in $\mathbb{P}\left([X_b]^\perp/[X_b]\right)$

h_α^{can} is Silverman's variations of canonical height on these sets

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G-dynamics on $\mathscr{X} \Leftrightarrow \Gamma$-dynamics on X (Cantat, L. Wang)

P-dynamics on $\mathscr{X} \Leftrightarrow \Gamma$-random walks on X

$P \subset G$ parabolic subgroup (Cantat—Dujardin)

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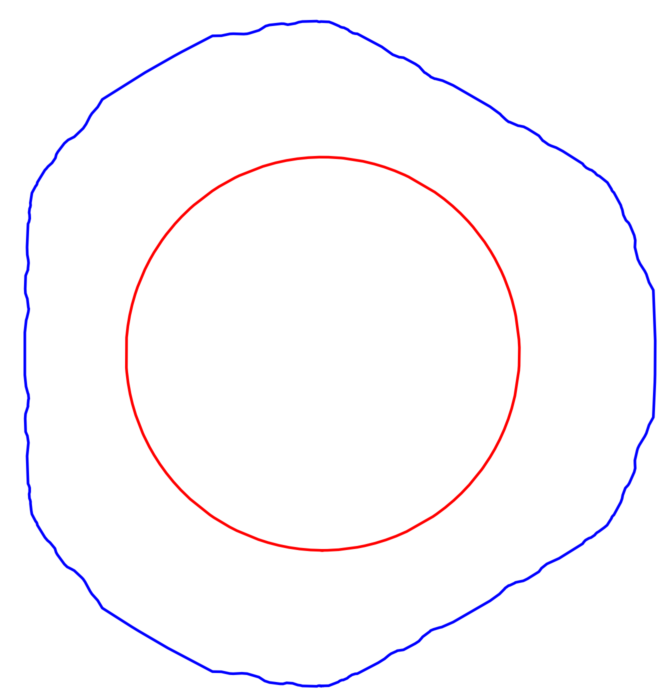
Invariants of $\mathrm{Aut}(X)$-orbits on $X(\overline{\mathbb{Q}})$:

volume of star-shaped set

(c.f. Silverman for a single automorphism)



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Precise counts of points ordered by height in $\mathrm{Aut}(X)$-orbits?

Thank you!