A couple of conjectures in arithmetic dynamics over fields of positive characteristic

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The Dynamical Mordell-Lang Conjecture

Throughout this talk, we let:

- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$;
- f^n denote the n-th iterate of the self-map f on some ambient space X (with $f^0 := \mathrm{id}_X$);

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- an arithmetic progression inside $\mathbb{N}_0$ is a set of the form $\{an + b\}_{n \in \mathbb{N}_0}$ for some given $a, b \in \mathbb{N}_0$ (so, in the case $a = 0$, we allow the arithmetic progression be a singleton).

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DML: *Given a quasiprojective variety X defined over a field K of characteristic 0 endowed with an endomorphism Φ, then for any subvariety $V \subseteq X$ and for any point $\alpha \in X(K)$, the set*

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is a finite union of arithmetic progressions.

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The next interesting case, still open for the DML conjecture is the case of arbitrary endomorphisms Φ of $\mathbb{A}^3$.

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For example, consider the case of the affine line $V \subset \mathbb{A}^2$ given by the equation $x + y = 1$ (over $\mathbb{F}_p(t)$) and the automorphism Φ of $\mathbb{A}^2$ given by

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Then the set S of all $n \in \mathbb{N}_0$ such that $\Phi^n(1, 1) \in V(\mathbb{F}_p(t))$ is the set

$$\{p^m \colon m \in \mathbb{N}_0\}$$

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One can construct other examples in which the return set S is even more complicated, as follows.

Another example

Let p be a prime number, let $V \subset \mathbb{G}_m^2$ be the curve defined over $\mathbb{F}_p(t)$ given by the equation $tx + (1-t)y = 1$, let $\Phi : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$ be the endomorphism given by
$\Phi(x,y) = \left(t^{p^2-1} \cdot x, (1-t)^{p^2-1} \cdot y\right)$, and let $\alpha = (1,1)$.

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Then the return set S of all $n \in \mathbb{N}_0$ such that $\Phi^n(\alpha) \in V$ is

$$\left\{ \frac{1}{p^2-1} \cdot p^{2n} - \frac{1}{p^2-1} : n \in \mathbb{N}_0 \right\}.$$

One more example

Let $p > 2$, let $K = \mathbb{F}_p(t)$, let $X = \mathbb{A}^3$, let $\Phi : \mathbb{A}^3 \longrightarrow \mathbb{A}^3$ given by $\Phi(x, y, z) = (tx, (1 + t)y, (1 - t)z)$, let $V \subset \mathbb{A}^3$ be the hyperplane given by the equation $y + z - 2x = 2$, and let $\alpha = (1, 1, 1)$.

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Then one can show that the return set S of all $n \in \mathbb{N}_0$ such that $\Phi^n(\alpha) \in V$ is

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All these examples motivate the following conjecture.

Dynamical Mordell-Lang Conjecture in positive characteristic

DML in characteristic p: *Given a quasiprojective variety* X *defined over a field* K *of characteristic* 0 *endowed with an endomorphism* Φ*, then for any subvariety* $V \subseteq X$ *and for any point* $\alpha \in X(K)$*, the set*

$$\{n \in \mathbb{N}_0 : \Phi^n(\alpha) \in V(K)\}$$

is a finite union of arithmetic progressions along with finitely many sets of the form

$$\left\{ \sum_{j=1}^{m} c_j p^{k_j n_j} \; : \; n_j \in \mathbb{N}_0 \text{ for each } j = 1, \ldots m \right\}, \tag{1}$$

for some $m \in \mathbb{N}$*, some* $c_j \in \mathbb{Q}$*, and some* $k_j \in \mathbb{N}_0$*.*

Results

Theorem (jointly with Pietro Corvaja, Thomas Scanlon and Umberto Zannier): *Let* $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ *be a regular self-map defined over a field* K *of characteristic* p*, let* $\alpha \in \mathbb{G}_m^N(K)$ *and let* $V \subseteq \mathbb{G}_m^N$ *be a subvariety. Then the Dynamical Mordell-Lang Conjecture holds in the following two cases:*

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(1) $\dim(V) \leq 2$.

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(1) $\dim(V) \leq 2$.

(2) Φ *is a group endomorphism and there is no nontrivial connected algebraic subgroup* $G \subseteq \mathbb{G}_m^N$ *such that an iterate of* Φ *induces an endomorphism of* G *that equals a power of the Frobenius.*

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(1) $\dim(V) \leq 2$.

(2) Φ *is a group endomorphism and there is no nontrivial connected algebraic subgroup* $G \subseteq \mathbb{G}_m^N$ *such that an iterate of* Φ *induces an endomorphism of* G *that equals a power of the Frobenius. In other words, if we write the action of* Φ *as* $\vec{x} \mapsto \vec{x}^A$ *for some* N*-by-*N *matrix with integer entries, then* A *has no eigenvalue which is multiplicatively dependent with respect to* p*.*

Strategy

Step 1: A regular self-map $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ is a composition of a translation with a group endomorphism $\vec{x} \longrightarrow \vec{x}^A$ (for some $A \in M_{N,N}(\mathbb{Z})$).

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$$\left\{ \prod_{j=1}^{m} \gamma_j^{p^{k_j n_j}} : n_j \in \mathbb{N}_0 \right\},$$

for some given $\gamma_j \in \mathbb{G}_m^N(\overline{K})$ and $k_j \in \mathbb{N}_0$.

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Step 3. We are left to determine the set of all $n \in \mathbb{N}_0$ such that $\Phi^n(\alpha) \in S \cdot H$, for a given F-set $S \cdot H$.

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Then there exists $N \in \mathbb{N}$, *there exists an algebraically closed field* K, *there exists an algebraic group endomorphism* $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$, *there exists* $\alpha \in \mathbb{G}_m^N(K)$ *and there exists a subvariety* $V \subset \mathbb{G}_m^N(K)$

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$$u_n = \sum_{i=1}^{m} c_i q^{n_i}, \tag{2}$$

for some $n_1, \ldots, n_m \in \mathbb{N}_0$.

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For example, a special case of this polynomial-exponential equation is

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which is open when $m > 5$. One still expects that the set of $n \in \mathbb{N}_0$ satisfying the general polynomial-exponential equation

$$u_n = \sum_{i=1}^{m} c_i p^{n_i}$$

is a finite union of arithmetic progressions along with finitely many sets of the form

$$\left\{ \sum_{j=1}^{\ell} d_j p^{k_j n_j} : \; n_j \in \mathbb{N}_0 \right\}$$

but when $m > 2$, the case of a general linear recurrence sequence $\{u_n\}$ is open.

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So, in order to prove the DML in characteristic p, we needed to employ the aforementioned technical hypotheses which guarantee that either

(1) $m \leq 2$ (this is the case when the dimension of the subvariety $V \subseteq \mathbb{G}_m^N$ is at most 2);or

(2) no characteristic root of the linear recurrence sequence $\{u_n\}$ is multiplicatively dependent with respect to p (this is the case when Φ is a group endomorphism corresponding to a matrix $A \in M_{N,N}(\mathbb{Z})$ whose eigenvalues are not multiplicatively dependent with respect to p).

Beyond tori

For a regular self-map Φ on an isotrivial semiabelian variety G, the strategy works identically, only that this time we obtain that the problem is equivalent with solving even more general polynomial-exponential equations of the form:

$$u_n = \sum_{i=1}^{m} c_i \lambda_i^{n_i},$$

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At the opposite spectrum, if G were an abelian variety defined over an algebraically closed field K which has trivial trace over $\overline{\mathbb{F}_p}$, then actually the DML problem in characteristic p is identical in methods and solution to the classical DML problem for abelian varieties (and in this case, the return set is simply a finite union of arithmetic progressions).

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For arbitrary semiabelian varieties, and more general, for arbitrary ambient varieties, the DML problem in characteristic p is expected to be at least as difficult as the classical DML conjecture.

The Zariski dense orbit conjecture

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- Φ is a group endomorphism of a commutative linear algebraic group.
- Φ is an endomorphism of a projective surface.

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(iii) Generally, the strategy in all known instances when the Zariski dense conjecture was proven is to assume that condition (B) does not hold (i.e., that Φ does not leave invariant a non-constant rational function) and then use the arithmetic of the ambient variety X combined with various information on the map Φ to prove the existence of a Zariski dense orbit.

The picture in positive characteristic

If X is any variety defined over $\mathbb{F}_p$, then there exists no non-constant rational function $f : X \dashrightarrow \mathbb{P}^1$ invariant under the Frobenius endomorphism $F : X \longrightarrow X$ (corresponding to the field automorphism $x \mapsto x^p$);

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This motivates the following conjecture.

Conjecture 1: Let X be a quasiprojective variety defined over an algebraically closed field K of characteristic p and let $\Phi : X \dashrightarrow X$ be a dominant rational self-map defined over K as well. Assume $\operatorname{trdeg}_{\overline{\mathbb{F}_p}} K \geq \dim(X)$. Then either there exists $\alpha \in X(K)$ whose orbit under Φ is well-defined and Zariski dense in X, or there exists a non-constant rational function $f : X \dashrightarrow X$ such that $f \circ \Phi = f$.

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Once again, the Frobenius endomorphism complicates the arithmetic dynamics question; we expect this is the only obstruction from obtaining the aforementioned dichotomy for the Zariski dense orbit conjecture.

Conjecture 2: *Let K be an algebraically closed field of positive transcendence degree over $\overline{\mathbb{F}_p}$, let X be a quasiprojective variety defined over K, and let $\Phi : X \dashrightarrow X$ be a dominant rational self-map defined over K as well.*

Conjecture 2: *Let K be an algebraically closed field of positive transcendence degree over $\overline{\mathbb{F}_p}$, let X be a quasiprojective variety defined over K, and let $\Phi : X \dashrightarrow X$ be a dominant rational self-map defined over K as well. Then one of the following three conditions must hold:*

(A) *There exists $\alpha \in X(K)$ whose orbit $\mathcal{O}_\Phi(\alpha)$ is Zariski dense in X.*

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   (3) *$(\tau^{-1} \circ \varphi \circ \tau)$ restricted to Y induces the Frobenius endomorphism F of Y, which corresponds to the field automorphism $x \mapsto x^q$.*

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(B) *There exists a non-constant rational function $f : \mathbb{G}_m^N \dashrightarrow \mathbb{P}^1$ such that $f \circ \Phi = f$.*

(C) *There exist positive integers m and r, a connected algebraic subgroup Y of $\mathbb{G}_m^N$ of dimension at least equal to 2 and a translation map $\tau_y : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ corresponding to a point $y \in \mathbb{G}_m^N(K)$ such that*

$$\left(\tau_y^{-1} \circ \Phi^m \circ \tau_y\right)|_Y = (F^r)|_Y, \tag{3}$$

where F is the usual Frobenius endomorphism of $\mathbb{G}_m^N$ induced by the field automorphism $x \mapsto x^p$.

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The next examples of regular self-maps Φ on $\mathbb{G}_m^3$ defined over $K := \mathbb{F}_p(t)$ will show the various instances of conditions (A)-(C) from our result.

Example 1. $\Phi(x_1, x_2, x_3) = (\beta_1 x_1, \beta_2 x_2, \beta_3 x_3)$ for some given $\beta_1, \beta_2, \beta_3 \in K$.

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Example 3. $\Phi(x_1, x_2, x_3) = \left(x_1^p, x_2^{p^2}, x_3^{p^3}\right)$ satisfies condition (A) always, i.e., there exists a Zariski dense orbit.

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Case 1. $\Phi : \mathbb{G}_m^N \longrightarrow \mathbb{G}_m^N$ is a dominant group endomorphism $\vec{x} \mapsto \vec{x}^A$ for a matrix $A \in M_{N,N}(\mathbb{Z})$ whose eigenvalues are not roots of unity.

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Examples for Case 1

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A similar argument works each time when the eigenvalues of the matrix A corresponding to the group endomorphism Φ (in arbitrary dimensions) has eigenvalues whose quotients do not have absolute value equal to 1.

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Now, if β and γ are multiplicatively independent, then the orbit of $(1, 1, 1, 1)$ under Φ is Zariski dense in $\mathbb{G}_m^4$.

Beyond tori

The same strategy employed in our proof of Theorem 1 (i.e., the case of a field K of transcendence degree at least equal to N) should extend with appropriate modification to the general case when we replace $\mathbb{G}_m^N$ by a split semiabelian variety G defined over a finite field.

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Finally, the general case in Conjectures 1 and 2 when X is an arbitrary variety is expected to be just as difficult as the general case in the classical Zariski dense conjecture.