Model theory of group actions on fields

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Algebraic Dynamics and Model Theory

- A model-theoretic approach to algebraic dynamics goes through a first-order theory of difference fields (ACFA).
- This approach was fruitful: results of Chatzidakis/Hrushovski, Medvedev/Scanlon, and others.
- Difference fields (inversive ones) are the same as actions of the group $\mathbb{Z}$ by field automorphisms.
- In this talk, we discuss the model theory of actions of arbitrary groups on fields.
- This is joint work with
  - Özlem Beyarslan: virtually free groups and torsion groups;
  - Daniel Hoffmann: finite groups.

G-fields as first-order structures

- We fix a group G. By a G-field, we mean a field together with a G-action by field automorphisms. Similarly, we have the notions of G-field extensions, G-rings, etc.
- A G-field is a first-order structure in the following way:

$$\mathsf{K} = (K, +, -, \cdot, 0, 1, g)_{g \in G}.$$

- Note that any g above denotes *three things* at the same time:
    - an element of G,
    - a function from K to K,
    - a formal function symbol.
- It is often convenient to consider the language where only a set of generators of G is specified. For example, difference fields have the first order structure: $(K, +, -, \cdot, 0, 1, \sigma)$, where σ may be understood as a chosen generator of $\mathbb{Z}$.

Existentially closed G-fields: definition

Let us fix a G-field K.

Systems of difference G-polynomial equations

Let $x = (x_1, \ldots, x_n)$ be a tuple of variables and $\varphi(x)$ be a system of difference G-polynomial equations over K:

$$\varphi(x) : \quad F_1(g_1(x_1), \ldots, g_n(x_n)) = 0, \ldots, F_n(g_1(x_1), \ldots, g_n(x_n)) = 0$$

for some $g_1, \ldots, g_n \in G$ and $F_1, \ldots, F_n \in K[X_1, \ldots, X_n]$.

Existentially closed G-fields

The G-field K is existentially closed (e.c.), if any system $\varphi(x)$ of difference G-polynomial equations over K which is solvable in a G-extension of K is already solvable in K.

Existentially closed G-fields: first properties

- Any G-field has an e.c. G-field extension (a general property of inductive theories).
- For $G = \{1\}$, the class of e.c. G-fields coincides with the class of algebraically closed fields (Hilbert's Nullstellensatz).
- For $G = \mathbb{Z}$, the class of e.c. G-fields coincides with the class of transformally (or difference) closed fields (models of ACFA).
- Any model of ACFA is algebraically closed. However, an e.c. G-field is usually not algebraically closed.
- The complex field $\mathbb{C}$ with the complex conjugation is not an e.c. C_2-field. (C_n denotes the cyclic group of order n.)

PAC fields and existentially closed G-fields

- For a G-field K, we usually denote by C its subfield of invariants K^G.
- If G is finitely generated, then C is a definable subfield of K, but in general there is no reason for that (it is merely type-definable).
- A field F is pseudo algebraically closed (PAC), if any absolutely irreducible variety over F has an F-rational point.
- If K is an e.c. G-field, then K is perfect PAC. If moreover G is finitely generated, then C is perfect PAC as well.

The theory G-TCF

Definition

If there is a first-order theory whose models are exactly e. c. G-fields, then we call this theory G-TCF and we say that G-TCF exists (G-TCF is a model companion of the theory of G-fields).

Example

- For $G = \{1\}$, we get G-TCF $=$ ACF.
- For $G = F_m$ (free group), we get G-TCF $= \mathrm{ACFA}_m$.
- If G is finite, then G-TCF exists (Sjögren, independently Hoffmann-K.)
- $(\mathbb{Z} \times \mathbb{Z})$-TCF does not exist (Hrushovski).

Axioms for ACFA

- Let (K, σ) be a difference field.
- By a variety, we mean an affine K-variety of finite type which is K-irreducible and K-reduced (i.e. a prime ideal in $K[\bar{X}]$).
- For any variety V, we also have the variety ${}^{\sigma}V$ and the bijection (not a morphism!)

$$\sigma_V : V(K) \to {}^{\sigma}V(K).$$

Geometric axioms for ACFA (Chatzidakis-Hrushovski)

(K, σ) is e.c. if and only if for any pair of varieties (V, W), if $W \subseteq V \times {}^{\sigma}V$ and the projections $W \to V, W \to {}^{\sigma}V$ are dominant, then there is $a \in V(K)$ such that $(a, \sigma_V(a)) \in W(K)$.

Axioms for ACFA and fields C, K

- If (K, σ) is e.c., then C and K are perfect PAC.
- It can be also shown that in such a case K is algebraically closed and C is pseudofinite ($\mathrm{Gal}(C) \cong \widehat{\mathbb{Z}}$).
- However, these two items above are not enough to imply that a difference field is e.c. (a model of ACFA).
- For example, there is $\sigma \in \mathrm{Aut}(\mathbb{Q}^{\mathrm{alg}})$ such that the difference field $(\mathbb{Q}^{\mathrm{alg}}, \sigma)$ satisfies these two items, but it is not a model of ACFA.
- In other words, ACFA is not "axiomatized by Galois axioms" (this phrase will be formally defined later).

Geometric axioms for G-TCF, G finite

Assume that $G = \{g_1, \ldots, g_e\}$ is a finite group and K is a G-field.

Geometric axioms for G-TCF (Hoffmann-K.)

K is e.c. if and only if for any pair of varieties (V, W): IF

- $W \subseteq {}^{g_1}V \times \ldots \times {}^{g_e}V$,
- all projections $W \to {}^{g_i}V$ are dominant,
- Iterativity Condition: for any i, we have ${}^{g_i}W = \pi_i(W)$, where

$$\pi_i : {}^{g_1}V \times \ldots \times {}^{g_e}V \to {}^{g_i g_1}V \times \ldots \times {}^{g_i g_e}V$$

  is the appropriate coordinate permutation;

THEN there is $a \in V(K)$ such that

$$((g_1)_V(a), \ldots, (g_e)_V(a)) \in W(K).$$

Galois axioms for G-TCF, G finite

If K is an e.c. G-field for a finite G, then we have the following.

- The fields K and C perfect PAC.
- The G-field K is strict that is the action of G on K is faithful.
- The restriction map:

$$\mathrm{res} : \mathrm{Gal}(C) \longrightarrow \mathrm{Gal}(K/C) = G$$

is a (universal) Frattini cover that is: if $\mathcal{G}_0$ is a proper closed subgroup of the profinite group $\mathrm{Gal}(C)$, then $\mathrm{res}(\mathcal{G}_0) \neq G$.

Theorem (Galois axioms; Sjögren, independently Hoffmann-K.)

Any G-field satisfying the conditions above is e.c.

Model-theoretic properties for G-TCF (G finite)

- Simplicity of the theory ACFA was crucial for the model-theoretic analysis and applications.
- Any G-field K is bi-interpretable with the pure field $C = K^G$.
- If K is an e.c. G-field, then C is supersimple of SU-rank 1.
- G-TCF is supersimple of SU-rank $e(= |G|)$.
- G-TCF and $\mathrm{Th}(C)$ have elimination of imaginaries in their languages with finitely many extra constants.

Our strategy

- Find a generalization of the known results about the model theory of actions of free groups/finite groups on fields.
- There is a natural class of groups for such a generalization: virtually free groups, that is groups having a finite index subgroup which is free.
- Our axiomatization here is in a way "doubly geometric":
    - the axioms are geometric themselves,
    - the axioms use the geometry underlying a given virtually free group (to be explained soon).

Bass-Serre theory

Theorem (Karrass, Pietrowski, and Solitar)

Let H be a finitely generated group. Then TFAE:

- *H is virtually free;*
- *H is isomorphic to the fundamental group of a finite graph of finite groups.*

Fundamental group of graph of groups

The above fundamental group can be obtained by successively performing the following operations applied to finite groups:

- finitely many free products with amalgamation;
- finitely many HNN extensions.

Main Theorem (axioms given by graph of finite groups)

Theorem (Beyarslan-K.)

If G is finitely generated and virtually free, then G-TCF exists.

Example (gluing the axioms along a graph of finite groups)

- We consider the simplest example of

$$G = C_2 * C_2 = \langle \sigma, \tau \rangle (\cong D_\infty = \mathbb{Z} \rtimes C_2).$$

- G-fields are exactly fields with two involutive automorphisms.
- Such $(K; \sigma, \tau)$ is e.c. iff for any pair of varieties (V, W) s.t.
  - $W \subseteq V \times {}^\sigma V \times {}^\tau V$,
  - the Zariski closure of the projection of W on $V \times {}^\sigma V$ satisfies the "C_2-axioms" and similarly with the projection on $V \times {}^\tau V$;

  there is $a \in V(K)$ such that $(a, \sigma_V(a), \tau_V(a)) \in W(K)$.

Absolute Galois group and simplicity

- For a group H, let $\widehat{H}$ be the profinite completion.
- For a profinite $\mathcal{H}$, let $\widetilde{\mathcal{H}} \to \mathcal{H}$ be the universal Frattini cover.
- A profinite group is small, if it has finitely many closed subgroups of a given finite index.

Theorem (Beyarslan-K.)

Let G be infinite, finitely generated, virtually free, and not free. Then, the profinite group $\ker(\widetilde{\widehat{G}} \to \widehat{G})$ is not small.

Results by Chatzidakis together with the theorem above imply that the "new theories" are not simple.

Theorem (Beyarslan-K.)

The theory G-TCF is simple if and only if G is finite or G is free.

NSOP_1 and conjectures

- Nick Ramsey suggested an argument to show that G-TCF is NSOP_1 ("not simple but still quite nice").
- This argument depends on a Galois-theoretic description of e.c. G-fields, which needs to be proven.
- Besides, we conjectured that for a finitely generated group G, G-TCF exists if and only if G is virtually free.
- It should be possible to show that if $\mathbb{Z} \times \mathbb{Z}$ embeds in G, then G-TCF does not exist.
- If $\mathbb{Z} \times \mathbb{Z}$ embeds in G, then G is not virtually free; but the opposite implication does not hold (Tarski monster, infinite Burnside groups).

When G is not finitely generated

- If G is not finitely generated, then a geometric axiomatization becomes problematic, since it is hard to control the full action of G in a first-order way.
- One way to deal with this problem is to hope that the following general theorem is applicable (good logical asymptotic behaviour).

Theorem

Let $T_1 \subseteq T_2 \subseteq \ldots$ be a chain of theories whose model companions, denoted T_n^*, form a chain $T_1^* \subset T_2^* \subseteq \ldots$ as well. Then $T^* := \bigcup_{n>0} T_n^*$ is a model companion of $T := \bigcup_{n>0} T_n$

Direct limit and logical limit

- Let us assume that $G = \bigcup G_n$ (for simplicity, an increasing union) and that each theory G_n-TCF exists.
- If $(G_n\text{-TCF})_n$ is an increasing chain, then we are done.

Example (explanations, time permitting, on last slide)

- These assumptions are satisfied for $\mathbb{Q} = \bigcup \frac{1}{n!}\mathbb{Z}$ (Medvedev) yielding the theory $\mathbb{Q}$ACFA(=$\mathbb{Q}$-TCF).
- These assumptions are satisfied for the Prüfer p-group $C_{p^\infty} = \bigcup_n C_{p^n}$ yielding the theory's C_{p^∞}-TCF.
- C_p^2-TCF $\not\subseteq C_{p^2}^2$-TCF and $C_{p^\infty}^2$-TCF does not exist.
- C_2-TCF $\not\subseteq C_6$-TCF but $C_{\mathbb{P}}$-TCF exists! $(C_{\mathbb{P}} := C_2 \oplus C_3 \oplus \ldots)$

Torsion groups: main theorem

Theorem (Beyarslan-K.)

Let $A = \bigcup A_i$ be a commutative torsion group (A_i: finite).

- *$A-\mathrm{TCF}$ exists if and only if for each prime p, the p-primary part of A is either finite or isomorphic with the Prüfer p-group.*
- *If the theory $A-\mathrm{TCF}$ exists, then it is strictly simple.*

A-TCF is axiomatised by Galois axioms saying about an A-field K:

1. the action of A on K is faithful;
2. K is a perfect field;
3. for each i, K^{A_i} is PAC;
4. for each i, we have:

$$\mathrm{Gal}\left(K^{A_i}\right) \cong \mathcal{G}_i,$$

where $(\mathcal{G}_i)_i$ is a fixed collection of small profinite groups.

Explanations about reducts

- For languages $L \subseteq L'$, L-theory T, and L'-theory T': $T \subseteq T'$ if and only if for all $M' \models T'$, we have $M'|_L \models T$.
- If G is finite and $K \models G - \mathrm{TCF}$, then $\mathrm{Gal}(K^G) \cong \widetilde{G}$.
- If $K \models C_{p^2} - \mathrm{TCF}$, then

$$\mathrm{Gal}\left(K^{C_{p^2}}\right) = \widetilde{C_{p^2}} = \mathbb{Z}_p = \widetilde{C_p};$$

$$\mathrm{Gal}\left(K^{C_p}\right) = p\mathbb{Z}_p \cong \mathbb{Z}_p.$$

- If $K \models C_{p^2}^2 - \mathrm{TCF}$, then

$$\mathrm{Gal}\left(K^{C_{p^2}}\right) = \widetilde{C_{p^2}^2} = \widehat{F}_2(p) = \widetilde{C_p^2}.$$

However, no proper closed subgroup of $\widehat{F}_2(p)$ of finite index is isomorphic to $\widehat{F}_2(p)$ (profinite Nielsen-Schreier formula).