Difference equations over fields of elliptic functions

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The conjecture of Loxton and van der Poorten

- $K = \bigcup_{s\in\mathbb{N}} \mathbb{C}(x^{1/s})$, $\widehat{K} = \bigcup_{s\in\mathbb{N}} \mathbb{C}((x^{1/s}))$ (Puiseux power series)
- $\sigma, \tau \in Aut(K)$: $\sigma(x) = x^p$, $\tau(x) = x^q$ ($p, q \in \mathbb{N}$ multiplicatively independent), extended to $\widehat{K}$

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Theorem (Adamczewski-Bell, 2017)

Let $f \in \widehat{K}$ satisfy the Mahler equations

$$\begin{cases} \sum_{i=0}^{n} a_i \sigma^{n-i}(f) = 0 \\ \sum_{i=0}^{m} b_i \tau^{m-i}(f) = 0 \end{cases}$$

with $a_i, b_i \in K$. Then $f \in K$.

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1. It follows that if $a_i, b_i \in \mathbb{C}(x)$ and $f \in \mathbb{C}((x))$ then $f \in \mathbb{C}(x)$.
2. The theorem "lives" on $\mathbb{G} = \mathbb{G}_{m,\mathbb{C}}$, $K = \mathbb{C}(\widetilde{\mathbb{G}})$ (universal covering), $\sigma, \tau \in End(\mathbb{G})$.

An additive analogue

- $K = \mathbb{C}(x)$, $\widehat{K} = \mathbb{C}((x))$
- $\sigma, \tau \in Aut(K) : \sigma(x) = px$, $\tau(x) = qx$ $(p, q \in \mathbb{C}^\times$ multiplicatively independent), extended to $\widehat{K}$.

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Theorem (Bézivin-Boutabaa, 1992)

Let $f \in \widehat{K}$ satisfy the difference equations

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with $a_i, b_i \in K$. Then $f \in K$.

1. Theorem "lives" on $\mathbb{G} = \mathbb{G}_{a,\mathbb{C}}$, $K = \mathbb{C}(\widetilde{\mathbb{G}})$, $\sigma, \tau \in End(\mathbb{G})$.
2. R.Schäfke and M.Singer (JEMS, 2019): a uniform treatment of both theorems, as well as of other similar results.
3. Adamczewski-Dreyfus-Hardouin-Wibmer (arXiv, October 2020): a remarkable strengthening.

An elliptic analogue

- $\Lambda \subset \mathbb{C}$ lattice, $K_\Lambda = \mathbb{C}(\wp(z,\Lambda), \wp'(z,\Lambda))$ field of Λ-elliptic functions.
- $K = \bigcup_{\Lambda \subset \Lambda_0} K_\Lambda = \mathbb{C}(\widetilde{\mathbb{G}})$ where $\mathbb{G} = \mathbb{C}/\Lambda_0$ elliptic curve, $\widehat{K} = \mathbb{C}((z))$.
- $p, q \in \mathbb{Z}$ multiplicatively independent, $\sigma, \tau \in Aut(K)$, $\sigma f(z) = f(pz)$, $\tau f(z) = f(qz)$, extended to $\widehat{K}$. Again, $\sigma, \tau \in End(\mathbb{G})$.

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Theorem (dS, 2020)

Suppose $(p,q) = 1$. Assume $f \in \widehat{K}$ satisfies the elliptic difference equations

$$\begin{cases} \sum_{i=0}^{n} a_i \sigma^{n-i}(f) = 0 \\ \sum_{i=0}^{m} b_i \tau^{m-i}(f) = 0 \end{cases}$$

with $a_i, b_i \in K$. Then $f \in R = K[z, z^{-1}, \zeta(z,\Lambda)]$ where $\zeta(z,\Lambda)$ (the Weierstrass zeta function) is a primitive of $\wp(z,\Lambda)$ for some $\Lambda \subset \Lambda_0$.

Remarks

1. Do not know if can relax $(p,q)=1$.
2. Theorem is optimal: any $f\in R$ satisfies simultaneously p- and q- elliptic difference equations.
3. May ask for a finer result: if coefficients are in K_Λ, for which $\Lambda'\subset\Lambda$ does $f\in R_{\Lambda'}=K_{\Lambda'}[z,z^{-1},\zeta(z,\Lambda')]$?

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- *Basic difference* I: proving that $f \in \mathbb{C}((x))$ is in $\mathbb{C}(x)$ goes by meromorphic continuation, since a function that is everywhere meromorphic (including at the boundary points) is rational. Proving $f \in R$, involves, besides meromorphic continuation to $\mathbb{C}$, issues of *periodicity.*
- *Basic difference* II: f need not be in K ! This is related to the existence of non-trivial vector bundles over $\mathbb{G} = \mathbb{C}/\Lambda_0$ which are invariant under pull-back by σ and τ (Atiyah's bundles, 1957). In the rational case, every vector bundle over $\mathbb{G} = \mathbb{G}_m$ or $\mathbb{G}_a$ is trivial.

Γ-difference modules

Let K be a field, $\Gamma \to Aut(K)$ a group action, $C = K^\Gamma$ the constant field.

Definition

A Γ-difference module over K is a finite dimensional vector space M over K, equipped with a semi-linear action of Γ, i.e. $\forall \gamma \in \Gamma$ a $\Phi_\gamma \in GL_C(M)$, s.t.

- $\Phi_\gamma(av) = \gamma(a)\Phi_\gamma(v)$ $(a \in K, v \in M)$
- $\Phi_{\gamma\delta} = \Phi_\gamma \circ \Phi_\delta$

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Example

In the three examples of $\mathbb{G} = \mathbb{G}_m$, $\mathbb{G}_a$, $\mathbb{C}/\Lambda_0$ we have $K = \mathbb{C}(\widetilde{\mathbb{G}})$, $\Gamma = \langle \sigma, \tau \rangle \simeq \mathbb{Z}^2$ ($\because p, q$ multiplicatively independent) and

$$M = Span_K \left\{ \sigma^i \tau^j f \right\} \subset \widehat{K}.$$

Simultaneous Mahler / difference / elliptic difference equations $\Leftrightarrow$ $\dim_K M < \infty$.

- The three theorems are derived from theorems stating that **under the given assumptions** M is "*degenerate*" in some sense.
- **Key point** (food for thought):

$$2 = \mathrm{rk}(\Gamma) > \mathrm{tr.deg.}(K/C) = 1.$$

- In the two rational cases "degeneracy" means $M = M_0 \otimes_{\mathbb{C}} K$ where M_0 is a $\mathbb{C}$-representation of Γ (in our case, a pair of *commuting* $\Phi_\sigma, \Phi_\tau \in GL(M_0)$) and the action of Γ is extended to M semi-linearly. We say that M can be *descended* from K to $\mathbb{C}$, or that it has an underlying $\mathbb{C}$-structure.
- In the elliptic case M is the elliptic (p,q)-difference module in the title of the lecture, and "degeneracy" will be a more subtle structure theorem (related to the above-mentioned Atiyah vector bundles).

Coordinates and matrices

Let $\Gamma = \langle \sigma, \tau \rangle \simeq \mathbb{Z}^2 \subset Aut(K)$ as in the three examples. Let M be a Γ-difference module, $e_1, \ldots, e_r$ a basis $/K$.

- $\Phi_\sigma(e_j) = \sum_{i=1}^r a_{ij} e_i$, $\Phi_\tau(e_j) = \sum_{i=1}^r b_{ij} e_i$
- Only condition: $\Phi_\sigma \circ \Phi_\tau = \Phi_\tau \circ \Phi_\sigma \iff \sigma(B)A = \tau(A)B$, $A^{-1} = (a_{ij})$, $B^{-1} = (b_{ij})$ (*Consistency condition*).
- Change of basis $\rightsquigarrow (A', B') = (\sigma(C)^{-1}AC, \tau(C)^{-1}BC)$ (*Gauge equivalence*).

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Corollary

The classification of Γ-difference modules over K is equivalent to the classification of consistent pairs (A, B) in $GL_r(K) \times GL_r(K)$ up to gauge equivalence. Equivalently, the non-abelian cohomology $H^1(\Gamma, GL_r(K))$ (a pointed set only!).

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- Replacing GL_r by a linear algebraic group G over K $\rightsquigarrow$ "*Γ-difference modules with G-structure" (e.g. orthogonal, symplectic, filtrations,...).* See R. Kottwitz "Isocrystals with additional structure", Comp.Math. 1985.

Recall either $\widehat{K} = \bigcup_{s\in\mathbb{N}} \mathbb{C}((x^{1/s}))$, $\sigma(x) = x^p$, $\tau(x) = x^q$ (Mahler case, $\mathbb{G} = \mathbb{G}_m$) or $\widehat{K} = \mathbb{C}((x))$, $\sigma(x) = px$, $\tau(x) = qx$ ($\mathbb{G} = \mathbb{G}_a$ or $\mathbb{C}/\Lambda_0$).

Γ-difference modules over $\widehat{K}$

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Theorem (Formal structure theorem)

Let M be a Γ-difference module over $\widehat{K}$. Then $M = M_0 \otimes_{\mathbb{C}} \widehat{K}$ for a Γ-invariant $\mathbb{C}$-vector space M_0. Equivalently, any consistent pair (A, B) is gauge-equivalent over $\widehat{K}$ to a commuting scalar pair (A_0, B_0).

1. Proof based on theory of Newton polygons and slopes: structure of modules over the twisted polynomial ring $\widehat{K}\left\langle \Phi, \Phi^{-1} \right\rangle$.
2. Mahler case: (A_0, B_0) unique up to conjugation.
3. Similar theorems for F-isocrystals, by Manin and Dieudonné...

Proof of the Loxton-van der Poorten conjecture

Let $K = \bigcup_{s\in\mathbb{N}} \mathbb{C}(x^{1/s}), \sigma(x) = x^p, \tau(x) = x^q$. Theorem of Adamczewski and Bell follows from:

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Sketch of proof:

- Let $t_0 = x$, $t_\infty = 1/x$, $t_1 = x - 1$, local parameters. For $i = 0, \infty, 1$ let $\widehat{\mathcal{O}}_i = \mathbb{C}[[t_i]]$, $\widehat{K}_i = \mathbb{C}((t_i))$. Let (A, B) be a consistent pair over K describing M in some basis. By the formal structure theorem, there are $C_i \in GL_r(\widehat{K}_i)$ such that

$$(\sigma(C_i)^{-1} A C_i, \tau(C_i)^{-1} B C_i) = (A_i, B_i) \in GL_r(\mathbb{C}) \times GL_r(\mathbb{C})$$

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  (for $i = 0, \infty$ we may have to replace x by $x^{1/s}$ first).
- By weak approximation, replacing (A, B) by a gauge-equivalent pair over K, may assume

$$C_i \in GL_r(\widehat{\mathscr{O}}_i).$$

- Estimates on formal Taylor expansion + local analyticity of A $\Rightarrow C_i$ analytic in $|t_i| < \varepsilon$.

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- Functional equation

$$C_i = A^{-1}\sigma(C_i)A_i,$$

gives meromorphic continuation of C_0 to $0 \le |x| < 1$, of C_∞ to $1 < |x| \le \infty$, of C_1 to $0 < |x| < \infty$. Note, for any $\varepsilon > 0$, the union of $\sigma^n(D(1,\varepsilon))$ is $\mathbb{P}^1 - \{0, \infty\}$.

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- $C_{01} = C_0^{-1}C_1$ meromorphic in $0 < |x| < 1$ and satisfies

$$A_0 C_{01} = \sigma(C_{01})A_1.$$

This forces C_{01} to be scalar, since Laurent expansions on annuli of analyticity will be supported on $p^n\mathbb{Z}$ for any n. Thus C_1 is analytic at 0. Similar argument on $C_{\infty 1} = C_\infty^{-1}C_1$ shows C_1 is meromorphic everywhere on $\mathbb{P}^1$, hence in $GL_r(K)$. QED

Elliptic (p,q)-difference modules of rank 1,2

Let $K = \bigcup_{\Lambda \subset \Lambda_0} K_\Lambda$, $\sigma f(z) = f(z/p)$, $\tau f(z) = f(z/q)$, $p, q \in \mathbb{N}$ multiplicatively independent.

Proposition (dS, CMB 2020)

For $a, b \in \mathbb{C}^\times$ let $M_1(a,b)$ be the module Ke where

$$\sigma(e) = a^{-1}e, \ \tau(e) = b^{-1}e.$$

Then every rank 1 elliptic (p,q)-difference module M is isomorphic to a unique $M_1(a,b)$. Equivalently, M has a unique $\mathbb{C}$ structure. Equivalently, $H^1(\Gamma, \mathbb{C}^\times) \simeq H^1(\Gamma, K^\times)$.

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In rank 2 this is already false. Let

$$\zeta(z,\Lambda) = \frac{\sigma'(z,\Lambda)}{\sigma(z,\Lambda)} \text{ (Weierstrass zeta function)}$$

$$\zeta'(z,\Lambda) = -\wp(z,\Lambda),\ \zeta(z+\omega,\Lambda) = \zeta(z,\Lambda)+\eta(\omega,\Lambda)\ (\omega\in\Lambda)$$

where η is the Legendre η-function.

Let

$$g_p(z,\Lambda) = p\zeta(qz,\Lambda) - \zeta(pqz,\Lambda),\ g_q(z,\Lambda) = q\zeta(pz,\Lambda) - \zeta(pqz,\Lambda).$$

Then $g_p, g_q \in K$. The matrices

$$A = \begin{pmatrix} 1 & g_p(z,\Lambda) \\ 0 & p \end{pmatrix},\ B = \begin{pmatrix} 1 & g_q(z,\Lambda) \\ 0 & q \end{pmatrix}$$

form a consistent pair, and we let M_2^{st} be the associated module:

$$M_2^{st} = K^2,\ \Phi_\sigma(v) = A^{-1}\sigma(v),\ \Phi_\tau(v) = B^{-1}\tau(v).$$

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Proposition

Every rank 2 elliptic (p,q)-difference module either admits a unique $\mathbb{C}$-structure or is isomorphic to $M_2^{st}(a,b) = M_2^{st} \otimes M_1(a,b)$ for unique $a, b \in \mathbb{C}^\times$.

The classification theorem: first steps

Let M be a rank r module over K, represented by the consistent pair (A, B) in some basis.

- By the formal structure theorem there exists $C \in GL_r(\widehat{K})$ such that $(\sigma(C)^{-1}AC, \tau(C)^{-1}BC) = (A_0, B_0)$ is a commuting pair of scalar matrices.

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- Let $D \in GL_r(K)$ be very close to C. Replacing (A, B) by the gauge-equivalent $(\sigma(D)^{-1}AD, \tau(D)^{-1}BD)$ and C by $D^{-1}C$ we may assume $C \in GL_r(\widehat{\mathscr{O}})$ where $\widehat{\mathscr{O}} = \mathbb{C}[[z]]$. Then A is analytic at 0.

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- Estimates on the formal Taylor expansion of C + analyticity of A at 0 $\Rightarrow$ C is analytic in $D(0, \varepsilon)$
- Functional equation $\sigma(C) = ACA_0^{-1}$ and the fact that $\bigcup \sigma^n(D(0, \varepsilon)) = \mathbb{C} \Rightarrow C$ is everywhere meromorphic on $\mathbb{C}$.

Unfortunately (or fortunately...) C need not be Λ-periodic for any Λ, as the rank 2 example above shows.

The periodicity theorem

- Let $\mathscr{M}$ be the sheaf of meromorphic functions on $\mathbb{C}$ (in the classical topology), $\mathscr{O}$ the sheaf of holomorphic functions,

$$\mathscr{G} = GL_r(\mathscr{M}),\ \mathscr{H} = GL_r(\mathscr{O}),\ \mathscr{F} = \mathscr{G}/\mathscr{H}.$$

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- Note: (1) $C \in \Gamma(\mathbb{C}, \mathscr{G})$ (2) $\mathscr{F}$ is a sheaf of cosets, its sections are discretely supported (i.e. $s \in \mathscr{F}(U) \Rightarrow \{\xi \in U | s_\xi \neq 0_\xi\}$ has no accumulation point in U) and (3) the stalk at each ξ,

$$\mathscr{F}_\xi = GL_r(\mathbb{C}((z-\xi)))/GL_r(\mathbb{C}[[z-\xi]])$$

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- We identify the stalk at ξ and the stalk at $\xi + \omega$ ($w \in \Lambda$) via translation. We call $s \in \Gamma(\mathbb{C}, \mathscr{F})$ Λ-periodic if $s_\xi = s_{\xi+\omega}$ for every $\xi \in \mathbb{C}$, $\omega \in \Lambda$. We denote by $\Gamma_\Lambda(\mathbb{C}, \mathscr{F})$ the Λ-periodic sections of $\mathscr{F}$.

- Let $\mathscr{M}$ be the sheaf of meromorphic functions on $\mathbb{C}$ (in the classical topology), $\mathscr{O}$ the sheaf of holomorphic functions,

$$\mathscr{G} = GL_r(\mathscr{M}),\ \mathscr{H} = GL_r(\mathscr{O}),\ \mathscr{F} = \mathscr{G}/\mathscr{H}.$$

- Note: (1) $C \in \Gamma(\mathbb{C}, \mathscr{G})$ (2) $\mathscr{F}$ is a sheaf of cosets, its sections are discretely supported (i.e. $s \in \mathscr{F}(U) \Rightarrow \{\xi \in U | s_\xi \neq 0_\xi\}$ has no accumulation point in U) and (3) the stalk at each ξ,

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- If $s \in \Gamma(\mathbb{C}, \mathscr{F})$ we call $s' \in \Gamma(\mathbb{C}, \mathscr{F})$ a *modification at* 0 of s if $s|_{\mathbb{C}-\{0\}} = s'|_{\mathbb{C}-\{0\}}$.

Theorem (Periodicity Theorem)

Assume $(p,q)=1$. *Let* $\overline{C}\in\Gamma(\mathbb{C},\mathscr{F})$ *be the image of* $C\in\Gamma(\mathbb{C},\mathscr{G})$. *Then there exists a modification of* $\overline{C}$ *at 0, denoted* s, *which is* Λ*-periodic, i.e.* $s\in\Gamma_{\Lambda}(\mathbb{C},\mathscr{F})$ *for some* $\Lambda\subset\Lambda_0$.

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Example. $r=1$, $\mathscr{F}=\mathscr{M}^\times/\mathscr{O}^\times \overset{\deg}{=} \mathbb{Z}$. Here $C(z)$ is a global meromorphic function such that $C(pz)/C(z)$ and $C(qz)/C(z)$ are both elliptic. The theorem says that a suitable modification at 0 of the *divisor* of C is periodic. In this case, by Abel-Jacobi we can infer that $z^m C(z)$ itself must be periodic for a suitable m.

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- Fix Λ, $\mathbb{A}_\Lambda=\prod'_{\xi\in\mathbb{C}/\Lambda}\widehat{K}_\xi\supset\mathbb{O}_\Lambda=\prod_{\xi\in\mathbb{C}/\Lambda}\widehat{\mathscr{O}}_\xi$ adeles of K_Λ
- $s\in\Gamma_\Lambda(\mathbb{C},\mathscr{F})=GL_r(\mathbb{A}_\Lambda)/GL_r(\mathbb{O}_\Lambda)$.
- C is determined by M only up to $C\rightsquigarrow DC$ with $D\in GL_r(K_\Lambda)\Rightarrow$ a well-defined

$$[s]\in Bun_{r,\Lambda}=GL_r(K_\Lambda)\setminus GL_r(\mathbb{A}_\Lambda)/GL_r(\mathbb{O}_\Lambda).$$

The vector bundle associated to M

- Recall $Bun_{r,\Lambda}$ classifies isomorphism classes of vector bundles of rank r on the elliptic curve $\mathbb{C}/\Lambda$.
- $M \rightsquigarrow C \rightsquigarrow s$ (periodic modification at 0 of $\overline{C}$) $\rightsquigarrow [s] = [\mathscr{E}_\Lambda]$
- Functoriality: If $\Lambda' \subset \Lambda$ then $pr^*_{\Lambda',\Lambda}(\mathscr{E}_\Lambda) \simeq \mathscr{E}_{\Lambda'}$. Also, $\mathscr{E}_\Lambda \simeq [p]^*(\mathscr{E}_\Lambda)$.

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Atiyah (1957) classified vector bundles on elliptic curves.

Theorem (Atiyah)

For each r there exists a unique vector bundle $\mathscr{F}_r$ on $\mathbb{C}/\Lambda$ which is indecomposable of rank r, has degree 0 and admits non-trivial global sections.

Proposition

(i) Given M, there exists a unique partition

$$(*)\ r = r_1 + r_2 + \cdots + r_k,\ \ r_1 \leq r_2 \leq \cdots \leq r_k$$

such that for all small enough Λ the vector bundle $\mathscr{E}_\Lambda$ is isomorphic to $\mathscr{F}_{r_1} \oplus \cdots \oplus \mathscr{F}_{r_k}$.
(ii) The vector bundle $\mathscr{F}_r$ corresponds to the class $[U_r] \in Bun_{r,\Lambda}$ where

$$U_r = \exp(\zeta(z,\Lambda)N_r)$$

and N_r is the nilpotent matrix with 1 in the $(i, i+1)$ entry $(1 \leq i \leq r-1)$ and 0 elsewhere.

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Call $(*)$ the *type* of M.

- M admits a $\mathbb{C}$-structure $\Leftrightarrow$ its type is $(1, 1, \ldots, 1)$
- From now on assume (to simplify the presentation) that the type of M is (r), i.e. $\mathscr{E}_\Lambda$ is indecomposable.

- $[s] = [U_r] \in Bun_{r,\Lambda}$ implies that, after a gauge transformation, we may assume

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Lemma (Key Lemma)

After conjugation by a scalar matrix commuting with U_r this forces

$$(2)\ T = a \cdot \mathrm{diag}[1, p, p^2, \ldots, p^{r-1}],\ S = b \cdot diag[1, q, q^2, \ldots, q^{r-1}]$$

for some $a, b \in \mathbb{C}^\times$.

Main Theorem

Theorem (Main Structure Theorem for type (r))

If the type of M is (r) then, up to a twist by $M_1(a, b)$, $M \simeq M_r^{st}$ where M_r^{st} corresponds to the consistent pair (A, B) given by (1) and (2).

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Final remarks.

1. The Key Lemma and the Periodicity Theorem are the main technical steps.
2. When the type is arbitrary, a more complicated structure theorem, but still completely explicit.
3. The theorem asserting that $f \in \widehat{K}$ satisfying simultaneously elliptic p- and q-difference equations lies in $R = K[z, z^{-1}, \zeta(z,\Lambda)]$ follows from the Main Structure Theorem applied to

$$M = Span_K(\sigma^i \tau^j f) \subset \widehat{K}.$$

Periodicity Theorem ($r = 1$)

- When r (the rank of M) is 1, the affine Grassmanian

$$\mathscr{F}_\zeta \simeq \mathbb{C}((z-\zeta))^\times / \mathbb{C}[[z-\zeta]]^\times \simeq \mathbb{Z}$$

is a group, and the Periodicity Theorem follows from:

Theorem

Let $s : \mathbb{R}^d \to \mathbb{Z}$ be a discretely supported function. Suppose $p, q \in \mathbb{N}$, $p, q \geq 2$, $(p, q) = 1$. If both $s_p(x) = s(px) - s(x)$ and $s_q(x) = s(qx) - s(x)$ are $\mathbb{Z}^d$-periodic, then after modifying s at 0 it becomes Λ-periodic for some lattice $\Lambda \subset \mathbb{Z}^d$.

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If s_p is any discretely supported $\mathbb{Z}^d$-periodic function

$$s(x) = \sum_{i=1}^{\infty} s_p(x/p^i)$$

is discretely supported, and satisfies $s_p(x) = s(px) - s(x)$, but need not be periodic.

The proof breaks into (i) periodicity on $\mathbb{Q}^d$ (ii) periodicity on $\mathbb{R}^d - \mathbb{Q}^d$, and uses different arguments in each case.

- Let S be a finite set of primes. For $x \in \mathbb{Z}$ write $x'_S = \prod_{p \in S} p^{-ord_p(x)} x$ (the S-deprived part of x). Fix $N \geq 1$ and say $x \sim_S y$ if $ord_p(x) = ord_p(y)$ for all $p \in S$ and also $x'_S \equiv y'_S \mod N$. The key to the case (i) is the following elementary Lemma.

Lemma

Let S and T be disjoint nonempty finite sets of primes, $N \geq 1$. Let $\sim$ be the equivalence relation on $\mathbb{Z}$ generated by $\sim_S$ and $\sim_T$. Then if x, y are non-zero, $x \sim y \Leftrightarrow x \equiv y \mod N$.

Key Lemma (r=2)

Need to study the consequences of the functional equation

$$A(z)U(z) = U(z/p)T(z)$$

where

$$A = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \ U = \begin{pmatrix} 1 & \zeta(z) \\ 0 & 1 \end{pmatrix}, \ T = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}$$

a, b, c, d Λ-elliptic, $\zeta(z) = \zeta(pqz, \Lambda)$, and $\alpha, \beta, \gamma, \delta$ everywhere holomorphic.

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- $\begin{pmatrix} & \\ & * \end{pmatrix} \rightsquigarrow$ Bootstrapping: $c\zeta(z) + d(z) = \delta(z) \Rightarrow c = \gamma = 0$ (take $\sum_{z \in \mathbb{C}/\Lambda} Res_z) \Rightarrow d(z) = \delta(z)$ constant.

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$$\zeta(z) - p\zeta(z/p) = g_p(z) = \beta(z) - b(z)$$

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- Higher r: same principles, only the algebra is more involved.

Thank you for your attention!

- The details can be found at

$$arXiv : 2007.09508$$

- Stay tuned for Hardouin's lecture on Friday!