

Fargues Bung  
 Wish to incarnate

$$D(S) \rightsquigarrow D(\mathcal{Q}\text{Coh}(M_{FG}))$$

⚡  
pairing with MU

Study  $M_{FG}$ , moduli stack of  
 1-dim'd formal groups.

Def  $M_{FG} \rightarrow \text{Spec}(\mathbb{Z})$  is the fpqc-  
 stack s.t.

$$M_{FG}(S) = \{ \underbrace{1\text{-dim'd formal Lie grps}}_{\sim} / S \}$$

formal grp. sch. Zar. locally / S  
 $\cong \hat{A}_S^1 = \text{Spf}(\mathcal{O}_S[[t]])$  //

Zariski presentation

$$\tilde{M}_{FG} = \text{Spec}(\Lambda)$$

⌞ Lazard ring

$M_{FG}$

$$\tilde{M}_{FG}(S) = \{ (g, \epsilon), \hat{A}_S^1 \xrightarrow{\sim} g \}$$

⚡  
1-dim. f. grp.

Localizing at  $p$ :

$M_{FG} \otimes \mathbb{Z}_p$  simpler presentation by  
moduli of  $p$ -typical formal grp.  
laws

$$\mathcal{G}/S \quad \hookrightarrow \quad \hat{A}_S^1 \xrightarrow{\sim} \mathcal{G}$$

s.t.  $\mathcal{L}$  seen as a curve  $\in$  Cartier  
modules  
is  $p$ -typical

$$F_n \cdot \mathcal{L} = 0 \quad \text{for } (n, p) = 1.$$

This presentation is given by

$$\text{Spec}(\mathbb{Z}_p[v_k \mid k \geq 0])$$

with universal formal group  
law given by the Cartier  
module with  $V$ -base  $\gamma$  and  
relation

$$F \cdot \gamma = \sum_{k \geq 0} V^k [v_k] \cdot \gamma.$$

Height stratification:

$(M_{FG} \otimes \mathbb{Z}_p)^{\leq h}$  open substack, where  
height  $\leq h$ . (Height  $h$  given  
by  $v_h \neq 0$ .)

Completion at  $p = \widehat{M}_{FG}$   $p$ -adic completion

$\text{Nil}_p = \text{Sch. on which } p \text{ is locally nilpotent}$

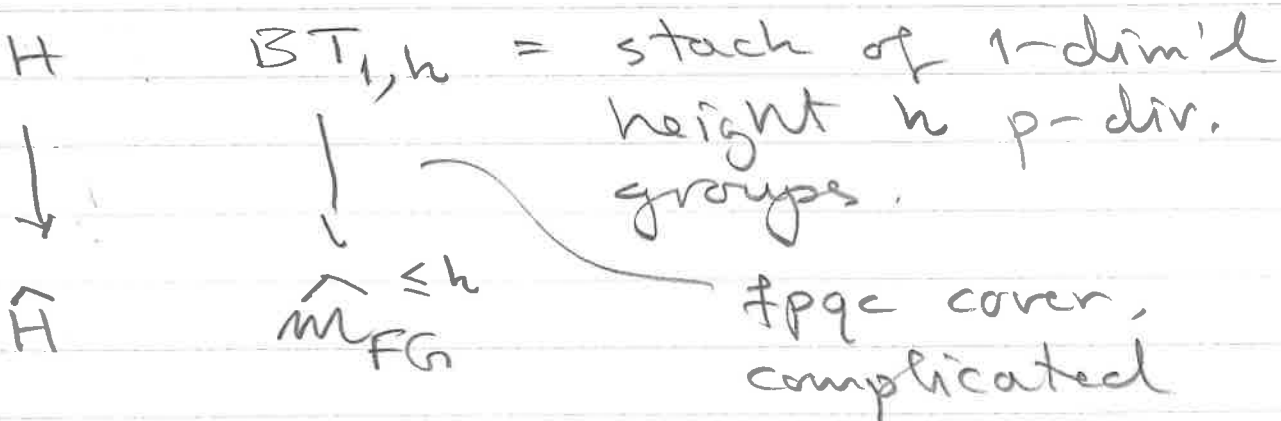
$S \longmapsto \{1\text{-dim'd f.g. / } S\}$

$$\widehat{M}_{FG}^{\leq h} \setminus \widehat{M}_{FG}^{\leq h-1} = [H / \mathcal{K}]$$

Here  $H / \text{Spf}(\mathbb{Z}_p)$  is a form of  $\mathcal{O}_D^* / \text{Spf}(\mathbb{Z}_p^h)$ , and  $\mathcal{K}$  is the Lubin-Tate space /  $\text{Spf}(\mathbb{Z}_p)$ .

$$\left( \begin{array}{ccc} \text{Gal}(\mathbb{F}_p^h / \mathbb{F}_p) & \longrightarrow & \text{Aut}_{\mathbb{Z}_p}(\mathcal{O}_D^*) \\ \text{Frob} & \longmapsto & (x \mapsto \pi x \pi^{-1}) \end{array} \right)$$

Another presentation:



Formal compl. kills étale part.

Consider generic fiber

$$(\widehat{\mathcal{M}}_{FG}^{\leq h})_{\eta} = v\text{-stack on } \text{Perf}_{\mathbb{Q}_p}$$

= perfectoid  $\mathbb{Q}_p$ -spaces  
in Scholze's  $v$ -top.

$$(R, R^+) \longmapsto \left\{ \begin{array}{l} \text{1-dim'd f.g. } / R^+ \\ \text{height } \leq h \end{array} \right\}$$

stack assoc. to this prestack.

Similarly, for  $BT_{1,h}$ , so get

$$(BT_{1,h})_{\eta}$$

$$\downarrow$$

$$(\mathcal{M}_{FG}^{\leq h})_{\eta}$$

given by HT  
period map

$$\underline{\text{Thm}} \quad (BT_{1,h})_{\eta} \simeq \left[ \underline{GL}_h(\mathbb{Q}_p) \backslash \mathbb{P}_{\mathbb{Q}_p}^{h-1, \diamond} \right]_{\cdot}$$

So obtain a  $v$ -covers

$$\Omega^{\diamond}$$

$$\subset \mathbb{P}_{\mathbb{Q}_p}^{h-1, \diamond}$$

$$\downarrow \simeq GL_h(\mathbb{Q}_p)\text{-inv.}$$

$$(\text{ht.} = h)$$

$$\subset (\widehat{\mathcal{M}}_{FG}^{\leq h})_{\eta}$$

open

and obtain

$$[\mathrm{GL}_h(\mathbb{Q}_p) \backslash \Omega^\diamond] \xrightarrow{\sim} \text{height } h \text{ part of } (\hat{M}_{FG}^{\leq h})_y$$

$\mathbb{R}$  — twin towers

$$[D^\star \backslash \mathbb{P}_{\mathbb{F}_p}^{h-1, \diamond}] \xrightarrow{\sim} \text{Gross-Hopkins}$$

(For  $\mathrm{GL}_2$ ,  $\Omega = \mathbb{P}^1_{\mathbb{R}} \setminus \mathbb{P}^1(\mathbb{Q}_p)$ .)

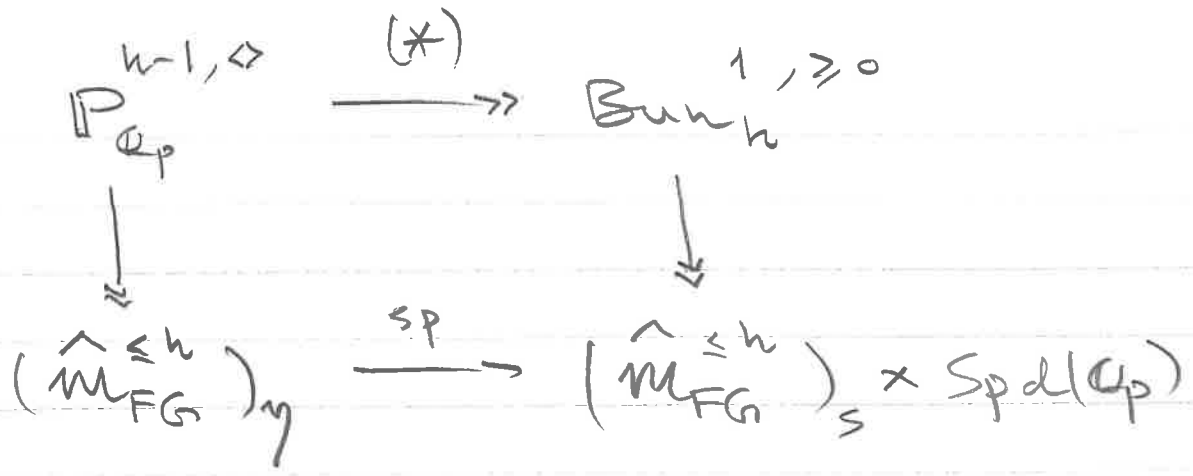
Second thm.: The spectral fiber  $(BT_{1,h})_s$  on  $\mathrm{Perf}_{\mathbb{F}_p}$ ,

$$(\mathbb{R}, \mathbb{R}^\star) \longmapsto \{ \text{1-dim'l f.g. } \mathbb{R}^\star \text{ height } \leq h \}$$

$$\text{Thm } (BT_{1,h})_s \xrightarrow{\sim} \mathrm{Bun}_h^1$$

Dieudonné functor

stack of degree 1 rank  $h$  vector bdl. on Fargues-Fontaine curve.

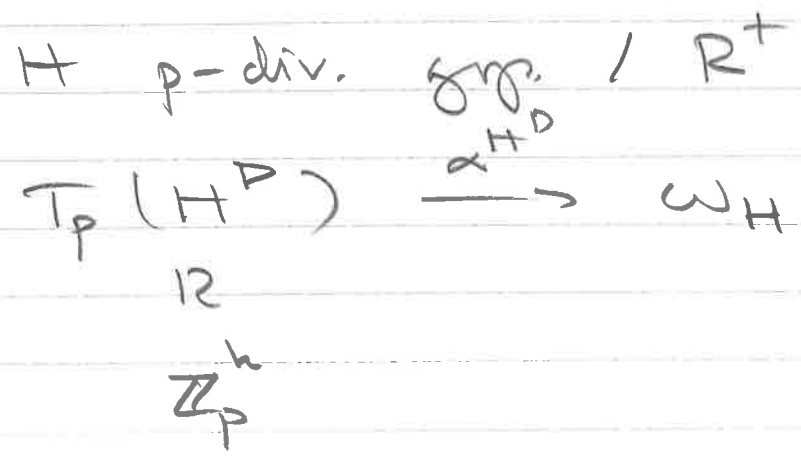


$(BT_{1,h})_{\eta}$  = moduli stack of rank  $h$  BKF-modules over  $\text{Ainf}$  s.t. coher  $(\varphi)$  is killed by  $\xi$ .

(\*) modification of degree 1 of  $\mathcal{O}^h$  / FF curve.

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Explain HT period map.



$$\mathbb{Q}_p / \mathbb{Z}_p \xrightarrow{x} H^D \longmapsto x^D : H \rightarrow \mu_{p^D} = \widehat{\mathbb{G}}_m$$

Note that, by Sen theory,

v.b. /  $\text{Spd}(\mathbb{Q}_p)$

$\xrightarrow{\sim} \text{Rep}_{\mathbb{F}_p}(\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p))$



f.d. semi-lin.  $\mathbb{F}_p$ -repr.

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So chromatic filtration corr.  
to Harder-Narasimhan filtr.

$$\text{Bun}_h = \text{Bun}_{\text{GL}_h}(\text{FF}_c).$$

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Third thm  $(\text{BT}_{1,h})_h$  is stack of BKF  
modules  $(M, \varphi)$  over  $\text{Ainf}(\mathcal{O}_c)$  of  
rk.  $h$  s.t.  $\text{coker}(\varphi)$  is annihilated  
by  $\mathfrak{f}$ ;  $\ker(\varphi) = (\mathfrak{f})$ .