Fargues Bun_G

Wish to incarnate

$$D(\mathbb{S}) \rightsquigarrow D(\mathrm{QCoh}(\mathcal{M}_{FG}))$$

pairing with MU

Study $\mathcal{M}_{FG}$, moduli stack of 1-dim'l formal groups.

Def $\mathcal{M}_{FG} \to \mathrm{Spec}(\mathbb{Z})$ is the fpqc-stack s.t.

$$\mathcal{M}_{FG}(S) = \{\text{1-dim'l formal Lie grps} / S\}$$

formal Lie grps $\sim$ formal grp. sch, Zar. locally $/S$ $\simeq \hat{\mathbb{A}}^1_S = \mathrm{Spf}(\mathcal{O}_S[[t]])$. //

Zariski presentation

$$\begin{array}{c} \tilde{\mathcal{M}}_{FG} = \mathrm{Spec}(\Lambda) \\ \downarrow \\ \mathcal{M}_{FG} \end{array}$$

Λ = Lazard ring

$$\tilde{\mathcal{M}}_{FG}(S) = \{(\mathcal{G}, \iota),\ \hat{\mathbb{A}}^1_S \xrightarrow{\sim} \mathcal{G}\}$$

$0 \longmapsto 0$

$\mathcal{G}$: 1-dim. f. grp.

Localizing at p:

$\mathcal{M}_{FG} \otimes \mathbb{Z}_p$ simpler presentation by moduli of p-typical formal grp. laws

$\mathcal{G}/S \quad \iota : \hat{\mathbb{A}}^1_S \xrightarrow{\sim} \mathcal{G}$

s.t. ι seen as a curve $\in$ Cartier modules is p-typical

$$F_n \cdot \iota = 0 \quad \text{for } (n,p) = 1.$$

This presentation is given by

$$\operatorname{Spec}(\mathbb{Z}_p[v_k \mid k \geq 0])$$

with universal formal group law given by the Cartier module with V-base γ and relation

$$F \cdot \gamma = \sum_{k \geq 0} V^k [v_k] \cdot \gamma .$$

Height stratification:

$(\mathcal{M}_{FG} \otimes \mathbb{Z}_p)^{\leq h}$ open substack, where height $\leq h$. (Height h given by $v_h \neq 0$.)

Completion at p: $\widehat{\mathcal{M}}_{FG}$ p-adic completion:

$$\begin{array}{ll} \mathrm{Nilp} : & \text{Sch. on which } p \text{ is locally nilpotent} \\ \quad \Downarrow & \\ \quad S & \longmapsto \{1\text{-dim'l f.g.} / S\} \end{array}$$

$$\widehat{\mathcal{M}}_{FG}^{\leq h} \setminus \widehat{\mathcal{M}}_{FG}^{\leq h-1} = [{}_H \backslash \mathfrak{X}]$$

Here $H / \mathrm{Spf}(\mathbb{Z}_p)$ is a form of $\mathcal{O}_D^* / \mathrm{Spf}(\mathbb{Z}_{p^h})$, and $\mathfrak{X}$ is the Lubin-Tate space $/ \mathrm{Spf}(\mathbb{Z}_p)$.

$$\left(\begin{array}{ccc} \mathrm{Gal}(\mathbb{F}_{p^h}/\mathbb{F}_p) & \longrightarrow & \mathrm{Aut}_{\mathbb{Z}_p}(\mathcal{O}_D^*) \\ \mathrm{Frob} & \longmapsto & (x \mapsto \pi x \pi^{-1}) \end{array} \right)$$

Another presentation:

$$\begin{array}{ccl} H & BT_{1,h} & = \text{stack of 1-dim'l height } h \text{ } p\text{-div. groups.} \\ \downarrow & \downarrow & \text{fpqc cover, complicated} \\ \hat{H} & \widehat{\mathcal{M}}_{FG}^{\leq h} & \end{array}$$

Formal compl. kills étale part.

Consider generic fiber

$(\widehat{\mathcal{M}}_{FG}^{\leq h})_\eta$ = v-stack on $\mathrm{Perf}_{\mathbb{Q}_p}$

= perfectoid $\mathbb{Q}_p$-spaces in Scholze's v-top.

$$(R, R^+) \longmapsto \left\{ \begin{array}{c} \text{1-dim'l f.g. } / R^+ \\ \text{height } \leq h \end{array} \right\}$$

$\quad$ ↯

stack assoc. to this prestack.

Similarly, for $BT_{1,h}$, so get

$$\begin{array}{c} (BT_{1,h})_\eta \\ \downarrow \\ (\mathcal{M}_{FG}^{\leq h})_\eta \end{array}$$

given by HT period map

Thm $(BT_{1,h})_\eta \simeq \left[\underline{GL_h(\mathbb{Q}_p)} \backslash \mathbb{P}_{\mathbb{Q}_p}^{h-1, \diamond} \right]$.

So obtain a v-covers

$$\begin{array}{ccc} \Omega^\diamond & \underset{\text{open}}{\subset} & \mathbb{P}_{\mathbb{Q}_p}^{h-1,\diamond} \\ \downarrow & & \downarrow \\ (ht. = h) & \underset{\text{open}}{\subset} & (\widehat{\mathcal{M}}_{FG}^{\leq h})_\eta \end{array}$$

$GL_h(\mathbb{Q}_p)$-inv.

and obtain

$$\left[GL_h(\mathbb{Q}_p) \backslash \Omega^\diamond \right] \xrightarrow{\sim} \text{height } h \text{ part of } (\hat{\mathcal{M}}_{FG}^{\leq h})_\eta$$

Ω twin towers

$$\left[D^\times \backslash \mathbb{P}^{h-1,\diamond}_{\mathbb{Q}_p} \right]$$

Gross-Hopkins

(For GL_2, $\Omega = \mathbb{P}^{1,an} \setminus \mathbb{P}^1(\mathbb{Q}_p)$.)

Second thm.: The special fiber $(BT_{1,h})_s$ on $\mathrm{Perf}_{\mathbb{F}_p}$,

$$(R, R^+) \longmapsto \{ \text{1-dim'l f.g. } / R^+ \text{ height } \leq h \}$$

<u>Thm</u> $(BT_{1,h})_s \xrightarrow{\sim} \mathrm{Bun}^1_h$

Dieudonné functor

stack of degree 1 rank h vector bdl. on Fargues-Fontaine curve.

$$\begin{tikzcd} \mathbb{P}^{h-1,\diamond}_{\mathbb{C}_p} \arrow[r, two heads, "(*)"] \arrow[d, "\approx"] & \mathrm{Bun}_h^{1,\geq 0} \arrow[d, two heads] \\ (\widehat{\mathcal{M}}^{\leq h}_{FG})_\eta \arrow[r, "sp"] & (\widehat{\mathcal{M}}^{\leq h}_{FG})_s \times \mathrm{Spd}(\mathbb{C}_p) \end{tikzcd}$$

$(BT_{1,h})_\eta$ = moduli stack of rk h BKF-modules over $A_{\inf}$ s.t. $\mathrm{coker}(\varphi)$ is killed by ξ.

(*) modification of degree 1 of $\mathcal{O}^h$ / FF curve.

Explain HT period map.

H p-div. grp. / R^+

$$\begin{array}{c} T_p(H^D) \xrightarrow{\alpha_{H^D}} \omega_H \\ \wr\wr \\ \mathbb{Z}_p^h \end{array}$$

$$\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{x} H^D \longmapsto x^D : H \longrightarrow \mu_{p^\infty} = \hat{\mathbb{G}}_m$$

Note that, by Sen theory,

$$\text{v.b.} \,/\, \mathrm{Spd}(\mathbb{C}_p) \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{C}_p}(\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p))$$

$$\wr$$

f.d. semi-lin. $\mathbb{C}_p$-repr.

So chromatic filtration corr. to Harder–Narasimhan filtr.

$$\mathrm{Bun}_h = \mathrm{Bun}_{GL_h}(FF_C).$$

<u>Third thm</u> $(BT_{1,h})_\eta$ is stack of BKF modules (M, φ) over $A_{inf}(\mathcal{O}_C)$ of rk. h s.t. $\mathrm{coker}(\varphi)$ is annihilated by ξ; $\ker(\theta) = (\xi)$.