The top-weight rational cohomology of A_g

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BIRS cohomology of arithmetic groups
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initiated at "Women in algebraic geometry" virtually at ICERM, co-organized with

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the project benefitted a lot from work of
Philippe Elbaz-Vincent, Herbert Gangl, Christophe Soulé
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A_g moduli space of principally polarized abelian varieties $/\mathbb{C}$
of dimension g.

$= \mathbb{H}_g / Sp(2g, \mathbb{Z})$.

$\mathbb{H}_g = \{ \tau \in M(g, \mathbb{C})^{sym} : \operatorname{Im} \tau > 0 \}$. $\quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$.

A_g is a variety, indeed a smooth separated Deligne-Mumford stack $\quad \dim A_g = \frac{(g+1)g}{2}$

$\mathcal{A}_g$ is a <u>variety</u>, indeed a smooth separated Deligne-Mumford stack $\quad \dim \mathcal{A}_g = \frac{(g+1)g}{2}$

For any variety X, $H^*(X;\mathbb{Q})$ carries a canonical <u>weight filtration</u> (Deligne)

$$W_0 \subset \cdots \subset W_{2i} = H^i(X;\mathbb{Q})$$

s.t. $\mathrm{Gr}^W_j H^i(X;\mathbb{Q}) = W_j / W_{j-1}$ has a pure Hodge structure of weight j.

compatible with Poincaré duality: for X smooth, $\quad d = \dim X$

$$\mathrm{Gr}^W_j H^i(X;\mathbb{Q}) \cong \left(\mathrm{Gr}^W_{2d-j} H_c^{2d-i}(X;\mathbb{Q})\right)^\vee.$$

$$\mathrm{Gr}^W_{2d} H^*(X;\mathbb{Q}) \cong \left(\mathrm{Gr}^W_0 H_c^{2d-*}(X;\mathbb{Q})\right)^\vee$$

"top weight cohomology" $\qquad$ weight 0 compactly supported.

Goal: study $H^*(\mathcal{A}_g;\mathbb{Q}) \cong H^*(Sp(2g,\mathbb{Z});\mathbb{Q})$

in fact, study $\mathrm{Gr}^W_{g^2+g} H^*(\mathcal{A}_g;\mathbb{Q})$.

<u>Thm</u> [BBCMMW]

$$\mathrm{Gr}^W_{30} H^k(\mathcal{A}_5;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k = 15, 20 \\ 0 & \end{cases}$$

deduced from computations of [Elbaz-Vincent, Gangl, Soulé]

$$\mathrm{Gr}^W_{42} H^k(\mathcal{A}_6;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k = 30 \\ 0 & \end{cases}$$

$$\mathrm{Gr}^W_{56} H^k(\mathcal{A}_7;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & k = 28, 33, 37, 42 \\ 0 & \end{cases}$$

$$\mathrm{Gr}^W_{72} H^k(\mathcal{A}_8;\mathbb{Q}) = 0 \qquad k \geq 60$$

deduced from vanishing results

$\mathrm{Gr}^W_{72} H^k(\mathcal{A}_8;\mathbb{Q}) = 0 \qquad k \geq 60$

$\mathrm{Gr}^W_{90} H^k(\mathcal{A}_9;\mathbb{Q}) = 0 \qquad k \geq 79$

$\mathrm{Gr}^W_{110} H^k(\mathcal{A}_{10};\mathbb{Q}) = 0 \qquad k \geq 99.$

deduced from vanishing results of [Dutour Sikiric, Elbaz-Vincent, Kupers, Martinet]

Remark $\mathrm{Gr}^W_6 H^k(\mathcal{A}_2;\mathbb{Q}) = 0$ [Igusa]

$\mathrm{Gr}^W_{12} H^k(\mathcal{A}_3;\mathbb{Q}) = \begin{cases} \mathbb{Q} & k=6 \\ 0 & \end{cases}$ [Hain '02: $H^6(\mathcal{A}_3;\mathbb{Q}) = E$

$0 \to \mathbb{Q}(-3) \to E \to \mathbb{Q}(-6) \to 0$].

$\mathrm{Gr}^W_{20} H^k(\mathcal{A}_4;\mathbb{Q}) = 0$ [Hulek-Tommasi]

Remark odd degree cohomology! [Grushevsky '10 "Geometry of $\mathcal{A}_g$ and its compactifications"] Open Problem 7.

Ingredients. [Ash-Mumford-Rapoport-Tai 75]

(1) The perfect cone/ first Voronoi toroidal compactification of $\mathcal{A}_g$ [Voronoi '08]

Consider all perfect forms $Q \in \mathrm{Sym}^2 \mathbb{R}^g$

Q determined up to scaling by its minimal vectors

Let $\sigma(Q) = \mathbb{R}_{\geq 0}\langle xx^T : x \in M(Q)\rangle$

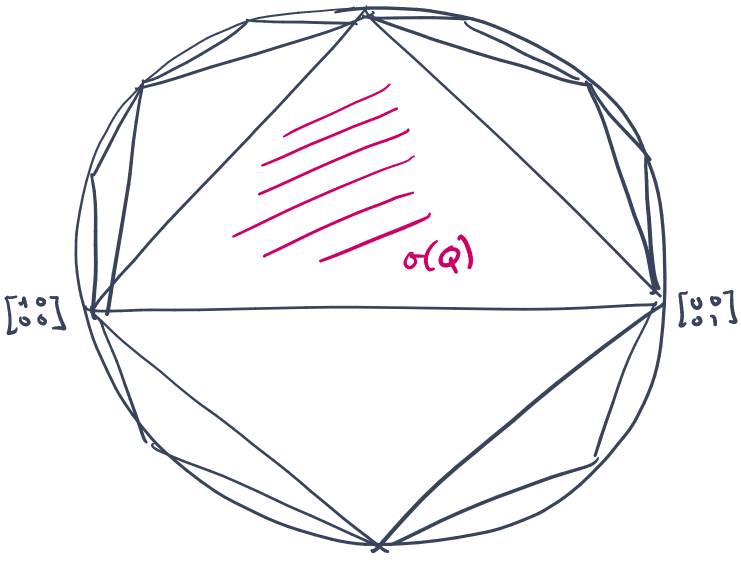
$M(Q) = \{x \in \mathbb{Z}^g \setminus \{0\} : Q(x) \leq Q(y)$ for all $y \in \mathbb{Z}^g \setminus \{0\}\}$.

$\bigcup_{Q \text{ perfect}} \sigma(Q)$ infinite polyhedral decomposition of

$\Omega_g^{rat} := \{X \in PSD_g : \ker X \text{ defined over } \mathbb{Q}\}$

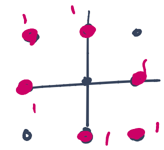
perfect cone decomposition $\qquad \mathcal{A}_g \subset \overline{\mathcal{A}}_g^{perf}$

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$



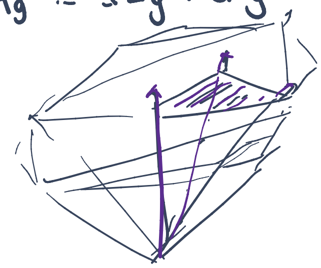
$g=2$

$Q(x,y) = x^2+xy+y^2$

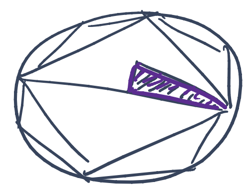
$M(Q) =$ 

$= \{\pm(1,0),\ \pm(0,1),\ \pm(1,-1)\}$

$A_g^{trop} := \Omega_g^{rat}/GL_g\mathbb{Z}$



$LA_g^{trop} = \left(\Omega_g^{rat}\setminus\{0\}/GL_g\mathbb{Z}\right)/\mathbb{R}_{>0}$



② Theorem [BBCMMW]

There is a rational chain complex $(P^{(g)}, \partial)$, spanned by equivalence classes of perfect cones s.t.

'perfect complex'

i) $$H_{i-1}(P^{(g)}) \cong \tilde{H}_{i-1}(LA_g^{trop};\mathbb{Q}) \cong Gr_{2d}^W H^{2d-i}(A_g;\mathbb{Q}).$$

Deligne

$A_g = \mathbb{H}_g/Sp(2g,\mathbb{Z})$

$p^{(0)} \subset \cdots \subset p^{(g-2)} \subset p^{(g-1)} \subset p^{(g)}$

ii) there is a short exact sequence,

$$0 \to p^{(g-1)} \longrightarrow p^{(g)} \longrightarrow V^{(g)} \longrightarrow 0$$

where $V^{(g)}$ is the Voronoi complex for $GL_g(\mathbb{Z})$, of Elbaz-Vincent - Gangl - Soulé.

$H_*(V^{(g)})$ computed for $g \le 7$ [EVGS]

③

$$H_m(V^{(g)}) \cong H_{m-g+1}(GL_g(\mathbb{Z}), St_g \otimes \mathbb{Q})$$

[EVGS, Soulé '00 "On the 3-torsion of $K_4(\mathbb{Z})$"]

[Lee-Szczarba '78]

[EVGS]

q \ p	0	1	2	3	4	5	6	7	8	9	10
21	0	0	0	0	0	0	0	$\mathbb{Q}$			
20	0	0	0	0	0	0	0	0			
19	0	0	0	0	0	0	0	0			
18	0	0	0	0	0	0	0	0			
17	0	0	0	0	0	0	0	0			
16	0	0	0	0	0	0	0	$\mathbb{Q}$			
15	0	0	0	0	0	0	0	0			
14	0	0	0	0	0	0	0	0			
13	0	0	0	0	0	0	0	0			
12	0	0	0	0	0	0	0	$\mathbb{Q}$			
11	0	0	0	0	0	0	0	0			
10	0	0	0	0	0	$\mathbb{Q}$	$\mathbb{Q}$	0			
9	0	0	0	0	0	0	0	0			
8	0	0	0	0	0	0	0	0			
7	0	0	0	0	0	0	0	$\mathbb{Q}$			
6	0	0	0	0	0	0	$\mathbb{Q}$	$\mathbb{Q}$			
5	0	0	0	0	0	$\mathbb{Q}$	$\mathbb{Q}$	0			
4	0	0	0	0	0	0	0	0	0		
3	0	0	0	$\mathbb{Q}$	$\mathbb{Q}$	0	0	0	0		
2	0	0	0	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

[DSEVKM]

$$E^1_{p,q} = \begin{cases} H_{p+q}(V^{(p)}) & 0 \le p \le 7 \\ 0 \end{cases}$$

$$\Rightarrow H_{p+q}(P^{(7)})$$

q \ p	0	1	2	3	4	5	6	7
21	0	0	0	0	0	0	0	$\mathbb{Q}$
20	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	$\mathbb{Q}$
15	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	$\mathbb{Q}$
11	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	$\mathbb{Q}$
6	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

$E^2_{\cdot,\cdot}$

[BBCMMW]

q \ p	0	1	2	3	4	5	6	7	8	9	10
21	0	0	0	0	0	0	0	$\mathbb{Q}$			
20	0	0	0	0	0	0	0	0			
19	0	0	0	0	0	0	0	0			
18	0	0	0	0	0	0	0	0			
17	0	0	0	0	0	0	0	0			
16	0	0	0	0	0	0	0	$\mathbb{Q}$			
15	0	0	0	0	0	0	0	0			
14	0	0	0	0	0	0	0	0			
13	0	0	0	0	0	0	0	0			
12	0	0	0	0	0	0	0	$\mathbb{Q}$			
11	0	0	0	0	0	0	0	0			
10	0	0	0	0	0	$\mathbb{Q}$	0	0			
9	0	0	0	0	0	0	0	0			
8	0	0	0	0	0	0	0	0			
7	0	0	0	0	0	0	0	$\mathbb{Q}$			
6	0	0	0	0	0	0	$\mathbb{Q}$	0			
5	0	0	0	0	0	$\mathbb{Q}$	0	0			
4	0	0	0	0	0	0	0	0	0		
3	0	0	0	$\mathbb{Q}$	0	0	0	0	0		
2	0	0	0	0	0	0	0	0	0	0	
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0

← vanishing implied by [Church-Farb-Putman '15, Church-Putman '16] [Lee-Szczarba]

& implies $\mathrm{Gr}^W_{g^2+g} H^{g^2-1}(A_g;\mathbb{Q})=0$

$$E_1^{p,q} = W_0 H_c^{p+q}(A_p;\mathbb{Q}) \Rightarrow \dots$$

[BBCMMW]

q \ p	0	1	2	3	4	5	6	7
21	0	0	0	0	0	0	0	$\mathbb{Q}$
20	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	$\mathbb{Q}$
15	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	$\mathbb{Q}$
11	0	0	0	0	0	0	0	0
10	0	0	0	0	0	$\mathbb{Q}$	0	0
9	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	$\mathbb{Q}$
6	0	0	0	0	0	0	$\mathbb{Q}$	0
5	0	0	0	0	0	$\mathbb{Q}$	0	0
4	0	0	0	0	0	0	0	0
3	0	0	0	$\mathbb{Q}$	0	0	0	0
2	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

← vanishing implied
← by [Church-Farb-Putman '15, Church-Putman '16]
& implies $Gr^W_{g^2+g} H^{g^2-1}(A_g;\mathbb{Q})=0$.

$$E_1^{p,q} = W_0 H_c^{p+q}(A_p;\mathbb{Q}) \Rightarrow W_0 H^{p+q}(A_p^{Satake};\mathbb{Q})$$

$A_g^{Sat} = A_g \sqcup A_{g-1} \sqcup \cdots \sqcup A_0$

q \ p	0	1	2	3	4	5	6	7
21	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	y_{14}
6	0	0	0	0	0	0	y_6^2	0
5	0	0	0	0	0	y_{10}	0	0
4	0	0	0	0	0	0	0	0
3	0	0	0	y_6	0	0	0	0
2	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

...

$H^*(A_\infty^{Sat};\mathbb{Q})$

$\Big\|$ [Charney-Lee '83]

$\mathbb{Q}[\underbrace{c_1, c_3, c_5, \ldots}_{\text{algebraic}}, \underbrace{y_6, y_{10}, y_{14}, \ldots}_{\text{weight } 0}]$

[Chen-Looijenga '15]

$E_\infty^{\cdot,\cdot}$ (guess! h/t Orsola)