Hermitian geometry of Oeljeklaus-Toma manifolds

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Theorem (Miyaoka, Todorov, Siu, Buchdahl, Lamari)

(M, J) compact complex surface admits a Kähler metric $\Leftrightarrow$ b_1 even.

Non-Kähler surfaces (known): Kodaira surfaces, properly elliptic surfaces, Inoue surfaces ($\mathcal{S}_A$, $\mathcal{S}^+$, $\mathcal{S}^-$), Hopf surfaces, Kato surfaces

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Today: Study Oeljeklaus-Toma manifolds (generalize Inoue surfaces of type $\mathcal{S}_A$)

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G_A be the group of affine transformations of $\mathbb{C} \times \mathbb{H}$ generated by:

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Theorem

Tricerri ('82) On S_A, the metric $\omega = \frac{dw \wedge d\overline{w}}{(\operatorname{Im} w)^2} + \operatorname{Im} w \, dz \wedge d\overline{z}$ is lcK ($d\omega = \frac{d \operatorname{Im} w}{\operatorname{Im} w} \wedge \omega$).

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$[K : \mathbb{Q}] = n = s + 2t$ (from now on, consider only the case $s, t \geq 1$). For any $s, t \in \mathbb{N}$, there exists $\mathbb{Q} \subseteq K$ with s real embeddings, $2t$ complex embeddings.

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$$\mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K \simeq \mathbb{R}^s_+ \times \mathbb{T}^n$$

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$$\mathcal{O}_K^{*,+} \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t$$

$$R_u(w_1, \ldots, w_s, z_{s+1}, \ldots, z_{s+t}) := (w_1 \cdot \sigma_1(u), \ldots, z_{s+t} \cdot \sigma_{s+t}(u)).$$

$$\mathcal{O}_K^{*,+} \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K$$

There exists a subgroup $U \subset \mathcal{O}_K^{*,+}$ of rank s such that the action

$$U \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K$$

is fixed-point-free, properly discontinuous, and co-compact.

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$$\mathcal{O}_K \rtimes U \circlearrowleft \mathbb{H}^s \times \mathbb{C}^t$$

Theorem (Oeljeklaus-Toma, 2005)

$$X(K, U) := \mathbb{H}^s \times \mathbb{C}^t / \mathcal{O}_K \rtimes U$$

is a compact complex manifold associated to algebraic number field K and to the admissible subgroup U of $\mathcal{O}_K^{*,+}$.

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What about the Hermitian geometry of $X(K, U)$?

Locally conformally metrics on $X(K, U)$

Theorem (Oeljeklaus, Toma, 2005, Battisti, 2014)

$X(K, U)$ admits a locally conformally Kähler metric if and only if

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  $t > s$ there are no loc. conf. Kähler metrics on $X(K, U)$.

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$$z^i = z_i + (\mathrm{Im}\ w_1)^{\frac{b_{1i}}{2}} \ldots (\mathrm{Im}\, w_s)^{\frac{b_{si}}{2}} e^{\mathrm{i} \sum_{k=1}^{s} c_{ki}\, \mathrm{Im}\ w_k} z'_i, \qquad 1 \leq i \leq t.$$

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We can represent

$$X(K, U) = U \ltimes \mathcal{O}_K \backslash \mathbb{R}^s \ltimes_\varphi (\mathbb{R}^s \times \mathbb{C}^t)$$

where

$$\varphi(x_1, \ldots, x_s) = \begin{pmatrix} \ddots & & & & \\ & e^{x_i} & & & \\ & & \ddots & & \\ & & & A_j & \\ & & & & \ddots \end{pmatrix},$$

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$$A_j := e^{\frac{1}{2}\sum_{k=1}^s b_{kj}x_k} \cdot e^{\mathrm{i}\sum_{k=1}^s c_{kj}x_k}$$

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Existence of special metric

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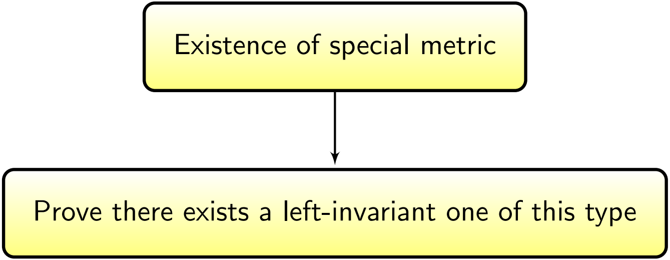

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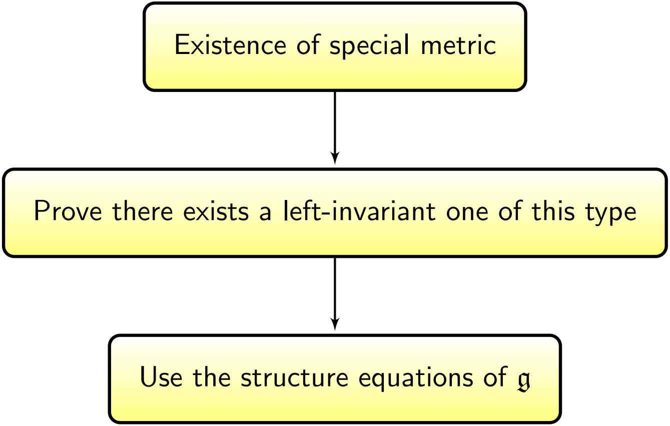

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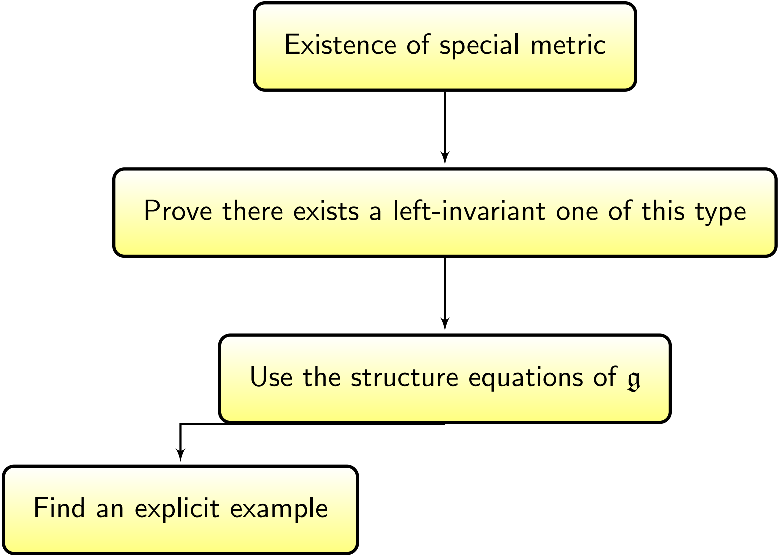

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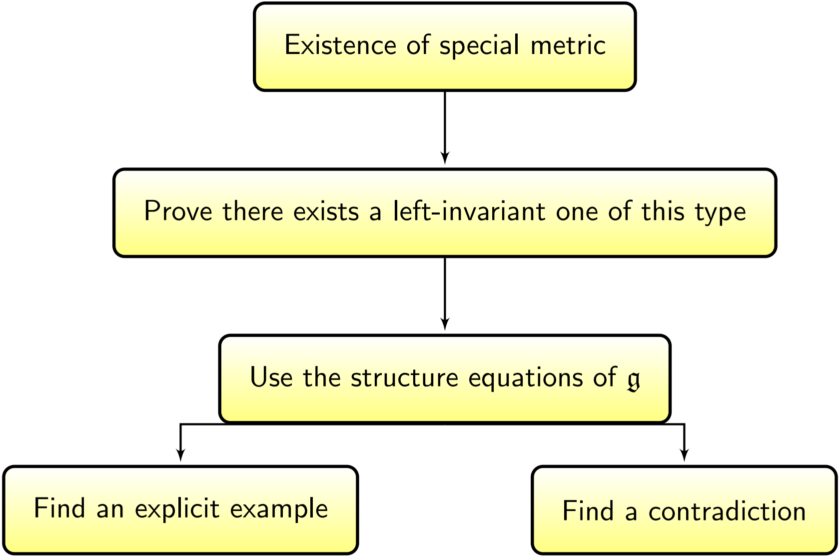

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- (Fino-Grantcharov, '04) If there exists a balanced metric on $\Gamma\backslash G$, there exists a left-invariant balanced metric Ω_0
- (Ugarte, '07) If there exists a pluriclosed metric on $\Gamma\backslash G$, there exists a left-invariant pluriclosed metric Ω_0

Definition

A metric Ω is balanced if $d\Omega^{n-1} = 0$, equivalently, if $d^*\Omega = 0$.

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- original averaging trick: Belgun, 2000, for lcK

Balanced and loc. conf. balanced metrics on $X(K, U)$

Theorem (-, 2020)

1. *An Oeljeklaus-Toma manifold $X(K, U)$ does not support balanced metrics.*
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1. The existence of a balanced metric $\rightsquigarrow \exists\ \Omega_0$ left-invariant balanced. Ω_0 balanced $\Leftrightarrow \Omega$ $(n-1, n-1)$-form positive & $d\Omega = 0$.
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- "(2) ⇒ (1)" Take:

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$b_1(X) = s = \frac{1}{2}\mathrm{dim}_{\mathbb{C}} X$

Theorem (D. Angella, A. Dubickas, -, J. Stelzig, '21)

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Remark: The loc. conf. Kähler and pluriclosed conditions are incompatible unless $X(K, U)$ is a surface.

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Proof.

Apply the number theoretical description of de Rham and Dolbeault cohomology (Istrati, -,2017, -,Toma, 2018) □

Corollary

Let $X(K,U)$ be an OT manifold of complex dimension 4. Then the following are equivalent:

1. *$X(K,U)$ admits a pluriclosed metric*
2. $b_3(X(K,U)) = 2$
3. $\dim_{\mathbb{C}} H^{2,1}_{\overline{\partial}} = 2$.

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Proof: No averaging trick in the astheno-Kähler case! Build a semi-positive $(2, 2)$ form $dd^c\eta \geq 0$. Then if Ω is astheno-Kähler,

$$0 \leq \int_X dd^c\eta \wedge \Omega^{n-2} = ... = \int_X \eta \wedge dd^c\Omega^{n-2} = 0.$$

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This is impossible: $x \mapsto \frac{\log(1-x)}{\log x}$ is strictly increasing on $(\frac{1}{2}, 1)$ and $\alpha \neq \alpha_1$. $X(K, U) \rightsquigarrow$ the first example in complex dimension 4 of pluriclosed OT manifold.

Thank you very much for your attention!