Higher representations and cornered Heegaard Floer homology

Andrew Manion (joint with Raphaël Rouquier)

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Gluing together Heegaard Floer invariants for surfaces

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Focus of this talk: in the case of Heegaard Floer homology, how do the categories for 2d surfaces behave under surface decompositions? $\leftrightarrow$ what can we assign to 1-manifolds, 0-manifolds?

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Basic definitions of HF already suggest the above, but it's realized most fully in bordered Heegaard Floer homology (Lipshitz–Ozsváth–Thurston '08; connection between LOT and Fukaya categories due to Auroux '10)

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HF invariants of genus-zero surfaces especially important when applying bordered HF ideas to compute knot Floer homology (HFK) in terms of tangle decompositions; in cornered HF one can ask how the algebra / category for multiple tangle endpoints arises from the algebra / category for a single tangle endpoint

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$$V \otimes V \otimes V^* \otimes V \otimes V \to V \otimes V \otimes V,$$

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$V =$ vector representation (2-dimensional)

$\otimes$: tensor product of representations of the quantum group (a Hopf algebra, so if V, W are representations then so is $V \otimes_k W$)

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A general construction $\bigotimes$ for categorified representations of Kac–Moody algebras is defined by Rouquier (in preparation)

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Theorem (M.–Rouquier '20)

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1. Explain a bit about $\otimes$ in the case where we define it
2. Discuss relationships to Heegaard Floer "strands algebras" and their gluing formulas as in Douglas–Manolescu

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$\mathbb{F}_2$-linear morphism spaces generated under composition and $\otimes$ by one endomorphism τ of e^2 with relations $\tau^2 = 0$ and $E\tau \circ \tau E \circ E\tau = \tau E \circ E\tau \circ \tau E$, differential $d(\tau) = 1$

2-representations

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Have version of $\bigotimes$ in both settings; second is most closely related to cornered Floer homology

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$\mathrm{Hom}_{\mathcal{U}}(\mathcal{V}_1, \mathcal{V}_2)$ should be a dg category with objects: dg functors $F : \mathcal{V}_1 \to \mathcal{V}_2$ "commuting with action of E," so $E_2 F \cong F E_1$ as dg functors (isomorphism or weaker notion of equivalence: will be vague here, just say "isomorphism")

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Thus: objects of $\mathrm{Hom}_{\mathcal{U}}(\mathcal{V}_1, \mathcal{V}_2)$ should be pairs (F, π)

Hom and internal Hom

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So: for 2-reps want to define $\mathbb{H}\mathrm{om}(\mathcal{V}_1, \mathcal{V}_2)$ (dg category with 2-action of $\mathcal{U}$) such that an object "vanishes" iff it's actually an object of $\mathrm{Hom}_{\mathcal{U}}(\mathcal{V}_1, \mathcal{V}_2)$

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Take $E(F, \pi)$ to be (m', π') where m' is the mapping cone of π (should assume $\mathcal{V}_1$ and $\mathcal{V}_2$ pretriangulated; will also assume idempotent complete): should have the right "vanishes iff π equivalence" when latter is made precise (won't do this)

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Can then guess at definitions of π', action of E on morphisms, and $\tau : E^2 \to E^2$, and show the construction works

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Generalize the above construction: let $\mathcal{W}$ be a dg category with endofunctors E_1, E_2 and natural endomorphisms τ_i of E_i^2 satisfying the usual relations (including differential)

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- $E(m, \pi) := (m', \pi')$ where m' is the mapping cone of π (makes sense in $\overline{W}^i$); can also define π', action of E on morphisms, and $\tau : E^2 \to E^2$

Examples of Δ_σ

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Object of $\Delta_\sigma(B\text{-mod})$: pair (m, π); define $\Delta_\sigma B$ so this is same data as a $\Delta_\sigma B$-module

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Let $E_1^\vee = \mathrm{Hom}_{B^{op}}(E_1, B)$ be the right dual (left adjoint) of E_1; then $\sigma : E_2E_1 \to E_1E_2$ is dual to a map $\lambda : E_1^\vee E_2 \to E_2E_1^\vee$ which we will also assume to be an isomorphism

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Define $\Delta_\sigma B := \frac{T_B^*(E_1^\vee \otimes_B E_2)}{(\ldots)}$ where the relation ideal is specified below (we'll just do tensor product case)

The relation ideal: tensor product case

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$$T_B^*(\mathcal{E}_1^\vee \otimes_B \mathcal{E}_2) \cong \bigoplus_{m=0}^{\infty} (E_1^\vee)^m \otimes_{\mathbb{F}_2} E_2^m$$

The relation ideal: tensor product case

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Define the relation ideal so that

$$A_1 \bigotimes A_2 := \Delta_\sigma B \cong \bigoplus_{m=0}^{\infty} (E_1^\vee)^m \otimes_{H_m} E_2^m,$$

$H_m :=$ endomorphism dg algebra of e^m in $\mathcal{U}$

Defining the bimodule E

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$$\pi : E_2 \otimes_{A_1 \otimes A_2} A_1 \bigotimes A_2 \to E_1 \otimes_{A_1 \otimes A_2} A_1 \bigotimes A_2$$

as bimodule over $(A_1 \otimes A_2, A_1 \bigotimes A_2)$; natural way to define left action of $A_1 \bigotimes A_2$ and endomorphism τ of E^2

Arc / chord diagrams

Now we'll discuss a key family of 2-representations of $\mathcal{U}$ on dg algebras: *strands algebras* $\mathcal{A}(\mathcal{Z})$ in bordered Heegaard Floer homology (first examples: Lipshitz–Ozsváth–Thurston '08)

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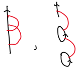
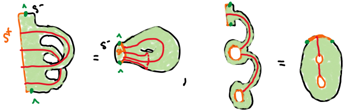
For us: $\mathcal{Z}$ will be an "arc diagram" (or "chord diagram") like those in Zarev '11: compact oriented 1-manifold $\mathcal{Z}$ with boundary (drawn in black) and 2-1 matching of finitely many points in interior of $\mathcal{Z}$ (drawn with red arcs)

Arc / chord diagrams (continued)

Examples: , taken to represent "sutured surfaces" (compact surfaces with boundary and extra data on boundary: "stopped regions" S_- and "unstopped regions" S_+ interfacing along a 0-manifold Λ of "sutures"



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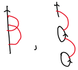
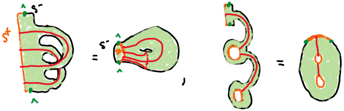
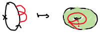
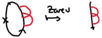
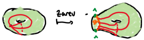
Examples: , taken to represent "sutured surfaces" (compact surfaces with boundary and extra data on boundary: "stopped regions" S_- and "unstopped regions" S_+ interfacing along a 0-manifold Λ of "sutures"



Compare: LOT's pointed matched circles $\mathcal{Z}$, taken to represent closed surfaces with basepoint : Zarev cuts open and views as chord diagram for corresponding surface with S^1 boundary and one stop on boundary





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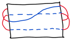
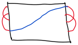
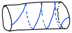
Basis over $\mathbb{F}_2$: strands pictures like   (including e.g. ) up to isotopy

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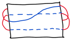
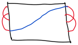
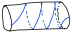
Basis over $\mathbb{F}_2$: strands pictures like   (including e.g. ) up to isotopy

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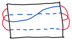
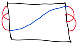
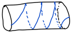
Basis over $\mathbb{F}_2$: strands pictures like   (including e.g. ) up to isotopy

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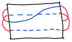
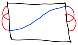
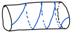
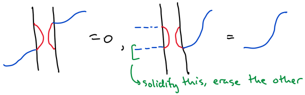
Basis over $\mathbb{F}_2$: strands pictures like   (including e.g. ) up to isotopy

- Drawn in $[0,1] \times \mathcal{Z}$
- Strands compatible with orientation
- No double-occupied matchings on right or left, except: any horizontal strands come in matched pairs (and are drawn dotted)

Strands algebras (continued)

Multiplication: concatenate, 

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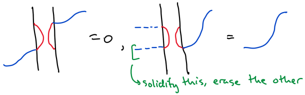

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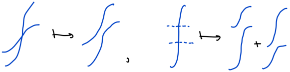

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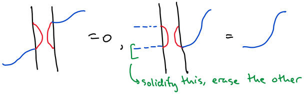
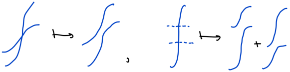
Multiplication: concatenate, 

Differential: 

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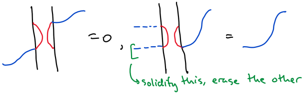
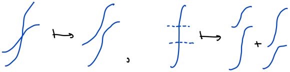
Multiplication: concatenate, 

Differential: 

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Heegaard Floer homology in general is based on Fukaya categories of these symmetric powers, explaining why these particular algebras are so natural for Heegaard Floer

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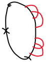
Douglas–Manolescu '11: asked how to recover algebra of e.g.  from data associated to , 

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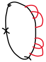
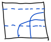
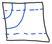
Douglas–Manolescu '11: asked how to recover algebra of e.g.  from data associated to , 

Their answer: associate certain algebraic constructions to top and bottom half, based on strands pictures like , 

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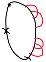
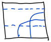
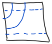
Douglas–Manolescu '11: asked how to recover algebra of e.g.  from data associated to , 

Their answer: associate certain algebraic constructions to top and bottom half, based on strands pictures like , 

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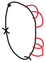
Douglas–Manolescu '11: asked how to recover algebra of e.g.  from data associated to , 

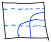
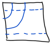
Their answer: associate certain algebraic constructions to top and bottom half, based on strands pictures like , 

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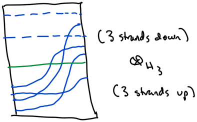
Illustration: 

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$E \otimes_{\mathcal{A}(\mathcal{Z})} \cdots \otimes_{\mathcal{A}(\mathcal{Z})} E$ (m factors) is isomorphic to the bimodule where m strands leave upward on distinguished component

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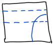
Dual $E^{\vee}$ of E is isomorphic to the bimodule where one strand leaves downward on distinguished component, e.g. 
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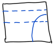
Dual $E^{\vee}$ of E is isomorphic to the bimodule where one strand leaves downward on distinguished component, e.g. 
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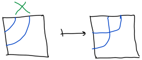
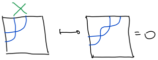
$E^2 = E \otimes_{\mathcal{A}(\mathcal{Z})} E$: have endomorphism τ sending ,


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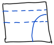
Dual $E^\vee$ of E is isomorphic to the bimodule where one strand leaves downward on distinguished component, e.g. 
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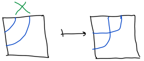
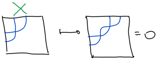
$E^2 = E \otimes_{\mathcal{A}(\mathcal{Z})} E$: have endomorphism τ sending ,


So: for each interval component of $\mathcal{Z}$, have a 2-action of $\mathcal{U}$ on $\mathcal{A}(\mathcal{Z})$

Expanding on the gluing formula

In our language:

Theorem (Douglas–Manolescu '11)

(For $\mathcal{Z}_i$ with one interval, in bijection with LOT's pointed matched circles): if $\mathcal{Z}$ is obtained by gluing $\mathcal{Z}_1$ and $\mathcal{Z}_2$ end-to-end, then $\mathcal{A}(\mathcal{Z}) \cong \mathcal{A}(\mathcal{Z}_1) \circledotimes \mathcal{A}(\mathcal{Z}_2)$ as dg algebras

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From this perspective: no more extra structure to consider on $\mathcal{A}(\mathcal{Z})$; extra structure comes from e.g.  but $\mathcal{A}(\mathcal{Z})$ comes from 

Expanding on the gluing formula (continued)

Theorem (M.–Rouquier '20)

(For any $\mathcal{Z}_i$ with distinguished interval components): if $\mathcal{Z}$ is obtained by gluing $\mathcal{Z}_1$ and $\mathcal{Z}_2$ end-to-end as above, then $\mathcal{A}(\mathcal{Z}) \cong \mathcal{A}(\mathcal{Z}_1) \circledast \mathcal{A}(\mathcal{Z}_2)$ as 2-representations of $\mathcal{U}$ (so E, τ also agree on both sides)

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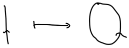
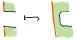
Theorem (M.–Rouquier '20)

(For any $\mathcal{Z}_i$ with distinguished interval components): if $\mathcal{Z}$ is obtained by gluing $\mathcal{Z}_1$ and $\mathcal{Z}_2$ end-to-end as above, then $\mathcal{A}(\mathcal{Z}) \cong \mathcal{A}(\mathcal{Z}_1) \bigotimes \mathcal{A}(\mathcal{Z}_2)$ as 2-representations of $\mathcal{U}$ (so E, τ also agree on both sides)

We also prove more involved version of this result for self-gluings like  based on a version of Δ_σ for 2-actions that "lax-commute" (in this case: we don't define 2-action on result, only for intervals rather than circles)

3 ways to view the gluing operation

What does this end-to-end gluing $\mathcal{Z}_1, \mathcal{Z}_2 \mapsto \mathcal{Z}$ look like on the sutured surfaces F_1, F_2, F that these chord diagrams represent? (At least) 3 equivalent ways to view it:

1. Glue small neighborhood of suture in ∂F_1 to small neighborhood of suture in ∂F_2: 

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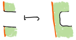
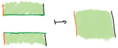
1. Glue small neighborhood of suture in ∂F_1 to small neighborhood of suture in ∂F_2: 
2. View sutured surfaces as cobordisms from S_- to S_+ restricting to id_Λ on the boundary; glue along an interval in id_Λ 

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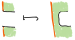
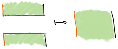
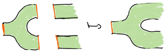
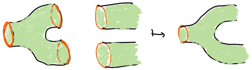
1. Glue small neighborhood of suture in ∂F_1 to small neighborhood of suture in ∂F_2: 
2. View sutured surfaces as cobordisms from S_- to S_+ restricting to id_Λ on the boundary; glue along an interval in id_Λ 
3. - Non-self-gluing case: glue "open pair of pants" to S_+ interval in F_1 and S_+ interval in F_2: 
   - Self-gluing case: glue  to S_+ interval in F

Open and closed pairs of pants

Compare the last interpretation with: tensor products for representations of (e.g.) $U_q(\mathfrak{sl}(2))$ and gluing *closed* pairs of pants: 

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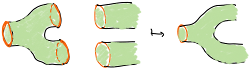
Compare the last interpretation with: tensor products for representations of (e.g.) $U_q(\mathfrak{sl}(2))$ and gluing *closed* pairs of pants: 

Related to: $\mathfrak{gl}(1|1)^+$ (intervals?) vs. $\mathfrak{gl}(1|1)$ (circles?); note that higher actions for circles are not apparent on the algebras $\mathcal{A}(\mathcal{Z})$ (need larger algebras?)

Ozsváth–Szabó's bordered HFK

Ozsváth–Szabó's algebras related to $\mathcal{Z} =$  are not n-fold end-to-end gluings like the ones considered here...

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These chord diagrams $\mathcal{Z}$: part of algebraic approach to HFK (work in preparation / progress) closely related to $\otimes$, similar in spirit to Ozsváth–Szabó's bordered HFK '16

Thanks

Thanks for your time!