The Alexander polynomial as a universal invariant

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Motivation for the universal quantum invariants

Introduced and studied (mostly in the context of finite dimensional or appropriately completed ∞ dimensional Hopf algebras) by Reshetikhin, Lawrence, Lee, Ohtsuki, Lyubashenko, Bruguières–Virelizier, Habiro, Virelizier, Murakami–Nagatomo, Willetts, . . .

- For a given Hopf algebra, all quantum invariants (semisimple and non semisimple) are encoded into a single representation independent algebraic object.
- Potential for better revealing the geometrical and topological significance of quantum invariants.

Today: construction on the basis of the **restricted dual** of a Hopf algebra.

Features

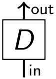
- Generality: universal quantum knot invariants from any (finite or ∞ dimensional) Hopf algebra with invertible antipode.
- Purely algebraic, no input topology (completion) is needed.
- Does not apply to links.

Outline

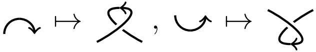
1. Reshetikhin–Turaev construction for long knots from rigid R-matrices (no ribbon element is used)
2. Adjunction between algebras and coalgebras
3. Drinfeld's quantum double
4. The universal quantum knot invariants
5. The Hopf algebra B_1 and its quantum double
6. The main result
7. Steps of the proof
8. Concluding remarks

1. Reshetikhin–Turaev construction for long knots from rigid R-matrices (no ribbon element is used)

An (oriented) **long knot diagram** D is a knot diagram in $\mathbb{R}^2$ with two open ends called "in" and "out":

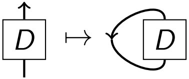
$D =$ 

Example: $D =$ 

The **normalization** of D is the long knot diagram $\dot{D}$ obtained from D by the replacements  .

D is called **normal** if $D = \dot{D}$.

The **building blocks** of normal long knot diagrams: four types of segments $\uparrow$, $\downarrow$,  ,  and eight types of crossings

Relation to closed oriented knot diagrams: 

An **R-matrix** over a finite-dimensional vector space V is an invertible linear map $r\colon V \otimes V \to V \otimes V$ that satisfies the quantum **Yang–Baxter relation**

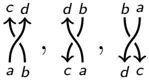
$$\hat{r}\check{r}\hat{r} = \check{r}\hat{r}\check{r}, \quad \hat{r} := r \otimes \mathrm{id}_V, \ \check{r} := \mathrm{id}_V \otimes r.$$

Let $B \subset V$ be a basis and $\{b^*\}_{b\in B} \subset V^*$ the dual basis defined by $\langle a^*, b\rangle = \langle a, b^*\rangle = \delta_{a,b}$. For any linear map $f\colon V \otimes V \to V \otimes V$, we associate its **partial transpose**

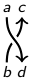
$$\widetilde{f}\colon V^* \otimes V \to V \otimes V^*, \quad \widetilde{f}(a^* \otimes b) = \sum_{c,d\in B} \langle a^* \otimes c^*, f(b \otimes d)\rangle c \otimes d^*.$$

An R-matrix r is called **rigid** if the linear maps $\widetilde{r^{\pm 1}}$ are invertible.

Let r be a rigid R-matrix over V with a basis B and D a normal long knot diagram with the set of edges E_D and the set of crossings C_D. A **local state** of D is a map $s\colon E_D \to B$. The **Bolzmann weight** of D in a state s: $\mathrm{W}_s(D) = \prod_{c\in C_D} \mathrm{W}_s(c)$ with



$$\xmapsto{\mathrm{W}_s} \langle c^*\otimes d^*, r(a\otimes b)\rangle,$$



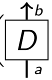
$$\xmapsto{\mathrm{W}_s} \langle a\otimes c^*, \left(\widetilde{r^{-1}}\right)^{-1}(b\otimes d^*)\rangle$$

and likewise for negative crossings with replacements $r \leftrightarrow r^{-1}$.

Theorem

Let normal D have equal number of negative and positive crossings. Then, the linear map $J_r(D)\colon V \to V$ defined by

$$\langle b^*, J_r(D)a\rangle = \mathrm{W}_{\partial s}\Bigg($$



$$\Bigg) := \sum_{\{s\colon E_D\to B|s(in)=a,\ s(out)=b\}} \mathrm{W}_s(D)$$

is an invariant of the (oriented) long knot represented by D.

2. Adjunction between algebras and coalgebras

The categories of algebras $\mathbf{Alg}_{\mathbb{K}}$ and coalgebras $\mathbf{Cog}_{\mathbb{K}}$ over a field $\mathbb{K}$ with two contravariant functors

$$(\cdot)^*\colon \mathbf{Cog}_{\mathbb{K}} \to \mathbf{Alg}_{\mathbb{K}} \text{ (dual (convolution) algebra)}$$

and

$$(\cdot)^o\colon \mathbf{Alg}_{\mathbb{K}} \to \mathbf{Cog}_{\mathbb{K}} \text{ (restricted dual coalgebra)}$$

($A^o \subset A^*$ is generated by all matrix coefficients of all finite dimensional representations of A).

$$\mathrm{Hom}_{\mathbf{Alg}_{\mathbb{K}}}(A, C^*) \simeq \mathrm{Hom}_{\mathbf{Cog}_{\mathbb{K}}}(C, A^o) \text{ (adjunction)}$$

The case of Hopf algebras: H^o is a Hopf algebra for any Hopf algebra H.

Example

- $C = \left(\mathbb{K}[\mathbb{Z}_{\geq 0}],\ \Delta(\chi_n) = \sum_{k=0}^{n} \chi_k \otimes \chi_{n-k}\right) \Rightarrow C^* \simeq \mathbb{K}[[x]]$
- $H = \left(\mathbb{C}[x],\ \Delta x = x \otimes 1 + 1 \otimes x\right) \Rightarrow H^o \simeq H \otimes \mathbb{C}[\mathbb{C}]$

3. Drinfeld's quantum double

Let $\mathbf{Hopf}_{\mathbb{K}}$ be the category of Hopf $\mathbb{K}$-algebras with invertible antipode. We have the contravariant endo-functor

$$(\cdot)^o \colon \mathbf{Hopf}_{\mathbb{K}} \to \mathbf{Hopf}_{\mathbb{K}} \text{ (restricted dual)}$$

Drinfeld's **quantum double** of a Hopf algebra $H \in \operatorname{Ob} \mathbf{Hopf}_{\mathbb{K}}$ is a Hopf algebra $D(H) \in \operatorname{Ob} \mathbf{Hopf}_{\mathbb{K}}$ determined (uniquely up to an isomorphism) by the property that there are two Hopf algebra inclusions

$$\imath \colon H \to D(H), \quad \jmath \colon H^{o,\mathrm{op}} \to D(H)$$

such that $D(H)$ is generated by their images subject to the commutation relations, $\forall (x, f) \in H \times H^o$,

$$(\jmath f)\imath x = \sum_{(x),(f)} \langle f_{(1)}, x_{(1)} \rangle \langle f_{(3)}, S(x_{(3)}) \rangle (\imath x_{(2)}) \jmath f_{(2)}$$

The restricted dual $D(H)^o$ of the quantum double is a **dual quasi-triangular** Hopf algebra with the **dual universal R-matrix**

$$\varrho\colon D(H)^o \otimes D(H)^o \to \mathbb{K}, \quad x \otimes y \mapsto \langle x, \jmath(\imath^o y) \rangle$$

which (among other properties) satisfies the Yang–Baxter relation

$$\varrho_{1,2} * \varrho_{1,3} * \varrho_{2,3} = \varrho_{2,3} * \varrho_{1,3} * \varrho_{1,2}$$

in the convolution algebra $((D(H)^o)^{\otimes 3})^*$.
Let $\{e_i\}_{i \in I} \subset H$ be a linear basis and $\{e^i\}_{i \in I} \subset H^*$ the associated set of canonical (dual) linear forms. Then, the formal universal R-matrix

$$R := \sum_{i \in I} \jmath e^i \otimes \imath e_i$$

is a formal conjugate of the dual universal R-matrix:

$$\langle x \otimes y, R \rangle = \langle \varrho, x \otimes y \rangle \quad \forall x, y \in D(H)^o.$$

4. The universal quantum knot invariants

For any finite-dimensional right co-module

$$V \to V \otimes D(H)^o, \quad v \mapsto \sum_{(v)} v_{(0)} \otimes v_{(1)},$$

one gets a rigid R-matrix

$$r_V : V \otimes V \to V \otimes V, \quad u \otimes v \mapsto \sum_{(u),(v)} v_{(0)} \otimes u_{(0)} \langle \varrho, v_{(1)} \otimes u_{(1)} \rangle.$$

The **universal quantum invariant** of long knots $Z_H(K)$ associated to a Hopf algebra $H \in \mathrm{Ob}\,\mathbf{Hopf}_{\mathbb{K}}$ takes its values in the convolution algebra $(D(H)^o)^*$ such that

$$J_{r_V}(K)v = \sum_{(v)} v_{(0)} \langle Z_H(K), v_{(1)} \rangle \quad \forall v \in V$$

where $J_{r_V}(K) \in \mathrm{End}(V)$ is the invariant of long knots associated to the rigid R-matrix r_V.

5. The Hopf algebra B_1 and its quantum double

$\mathrm{B}_1 = \mathbb{C}[a^{\pm 1}, b]$ with the co-products

$$\Delta(a) = a \otimes a, \quad \Delta(b) = a \otimes b + b \otimes 1$$

It is the algebra of regular functions on the affine linear algebraic group

$$\mathrm{Aff}_1(\mathbb{C}) := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right) \mid a \in \mathbb{C}_{\neq 0},\ b \in \mathbb{C} \right\}$$

The group structure of $\mathrm{Aff}_1(\mathbb{C})$ induces a commutative but non co-commutative Hopf algebra structure of B_1.

The restricted dual $\mathsf{B}_1^{o,\mathrm{op}}$ is composed of two Hopf sub-algebras: the group algebra $\mathbb{C}[\mathrm{Aff}_1(\mathbb{C})]$ generated by grouplike elements

$$\{\chi_{u,v} \mid (u,v) \in \mathbb{C} \times \mathbb{C}_{\neq 0}\}, \quad \chi_{u,v}\chi_{u',v'} = \chi_{u+vu',vv'},$$

and the universal enveloping algebra $U(\mathrm{Lie}\,\mathrm{Aff}_1(\mathbb{C}))$ generated by primitive elements ψ and ϕ satisfying $\phi\psi - \psi\phi = \phi$.

Interaction between $\mathbb{C}[\mathrm{Aff}_1(\mathbb{C})]$ and $U(\mathrm{Lie}\,\mathrm{Aff}_1(\mathbb{C}))$ in $\mathsf{B}_1^{o,\mathrm{op}}$

$$[\chi_{u,v}, \psi] = u\phi\chi_{u,v}, \quad \chi_{u,v}\phi = v\phi\chi_{u,v} \quad \forall (u,v) \in \mathbb{C} \times \mathbb{C}_{\neq 0}$$

Pairing between $\mathsf{B}_1^{o,\mathrm{op}}$ and B_1, $p = p(a,b) \in \mathbb{C}[a^{\pm 1}, b] = \mathsf{B}_1$

$$\langle \chi_{u,v}, p \rangle = p\Big(\frac{1}{v}, \frac{u}{v}\Big), \quad \langle \psi, p \rangle = \frac{\partial p}{\partial a}(1,0), \quad \langle \phi, p \rangle = \frac{\partial p}{\partial b}(1,0)$$

Interaction between B_1 and $\mathsf{B}_1^{o,\mathrm{op}}$ in $D(\mathsf{B}_1)$

$$[\psi, b] = b, \quad [\phi, b] = 1 - a, \quad \chi_{u,v}^{-1} b \chi_{u,v} = bv + (a-1)u$$

and a is a grouplike central element.

Remark

- The center $\mathcal{Z}(D(\mathsf{B}_1))$ is generated by the elements a and $c := \phi b + (a-1)\psi$.
- The Heisenberg relation $[\phi, b] = 1 - a$ with central a implies that, in any irreducible finite dim. representation of $D(\mathsf{B}_1)$, one has $a = 1$. In general, $a - 1$ should be nilpotent.
- The (formal) universal R-matrix

$$R = (1 \otimes a)^{\psi \otimes 1} e^{\phi \otimes b} = \sum_{m,n \geq 0} \frac{1}{n!} \binom{\psi}{m} \phi^n \otimes (a-1)^m b^n$$

trivialises to $e^{\phi \otimes b}$ at $a = 1$ with commuting ϕ and b. Thus, there are no non-trivial invariants from irreps.

6. The main result

Theorem

The universal quantum invariant of a long knot K associated to the Hopf algebra B_1 is of the form

$$Z_{\mathsf{B}_1}(K) = (\Delta_K(a))^{-1} \in \mathbb{C}[[a-1]] \subset (D(\mathsf{B}_1)^o)^*$$

where $\Delta_K(t)$ is the (canonically normalised) Alexander polynomial of K.

7. Steps of the proof

(a) The determinantal formula

Theorem (Kaufmann–Saleur, 1991)

Let a knot K be the closure of a braid $\beta \in B_n$ and $\psi_n(\beta) \in \mathrm{GL}_n(\mathbb{Z}[t^{\pm 1}])$ the image of β under the unrestricted Burau representation. Let $\hat{\beta}_n$ be obtained from $\psi_n(\beta)$ by throwing away the n-th column and the n-th row. Then,

$$\Delta_K(t) = t^{\frac{1-n-g(\beta)}{2}} \det(I_{n-1} - \hat{\beta}_n)$$

where I_k denotes the identity $k \times k$ matrix and $g\colon B_n \to \mathbb{Z}$ is the group homomorphism that sends the Artin generators to 1.

(b) Coherent states and the Gaussian integration formula

$H^n \subset L^2(\mathbb{C}^n, \mu_n)$ the Hilbert sub-space of holomorphic functions

$$\langle f|g\rangle := \int_{\mathbb{C}^n} \overline{f(z)} g(z)\, \mathrm{d}\mu_n(z), \quad \frac{\mathrm{d}\mu_n}{\mathrm{d}\lambda_{2n}}(z) = \frac{1}{\pi^n} e^{-\|z\|^2}$$

Schrödinger's coherent states $\varphi_u \in H^n$, $\varphi_u(z) = e^{u^\top z}$, $u \in \mathbb{C}^n$.
Reproducing property

$$\langle \varphi_{\bar{u}}|f\rangle = \int_{\mathbb{C}^n} \varphi_u(\bar{z}) f(z)\, \mathrm{d}\mu_n(z) = f(u) \quad \forall f \in H^n$$

$A^n \subset H^n$ the subspace generated by products of coherent states and polynomials.
Gaussian integration formula ($z^* := \bar{z}^\top$)

$$\int_{\mathbb{C}^n} e^{v^* z + z^* u + z^* M z}\, \mathrm{d}\mu_n(z) = \frac{e^{v^* W^{-1} u}}{\det(W)}, \quad W := I_n - M$$

(c) Representations of $D(\mathrm{B}_1)$ in $A^1[[\hbar]]$

Homomorphism of algebras $\rho_\lambda\colon D(\mathrm{B}_1) \to \mathrm{End}(A^1[[\hbar]])$

$$(a, b, \phi, \psi) \mapsto \left(1+\hbar, \frac{\partial}{\partial z}, \hbar z, \lambda - z\frac{\partial}{\partial z}\right), \quad \chi_{u,v} f(z) = e^{\hbar u z} f(vz),$$

$$\rho_\lambda^{\otimes 2}(R) = (1+\hbar)^\lambda \rho_0^{\otimes 2}(R) \in \mathrm{End}(A^1)^{\otimes 2}[[\hbar]],$$

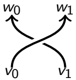
$$\rho_0^{\otimes 2}(R) = (1+\hbar)^{-z_0 \frac{\partial}{\partial z_0}} e^{\hbar z_0 \frac{\partial}{\partial z_1}} = \sum_{m,n\geq 0} \frac{\hbar^{m+n}}{n!} \binom{-z_0 \frac{\partial}{\partial z_0}}{m} \left(z_0 \frac{\partial}{\partial z_1}\right)^n$$

Action of $r := \rho_0^{\otimes 2}(R)P$ on the coherent states

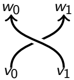
$$r\varphi_v = \varphi_{Uv}, \quad U := \begin{pmatrix} \frac{\hbar}{1+\hbar} & \frac{1}{1+\hbar} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}, \quad t := \frac{1}{1+\hbar},$$

This is the building block of the Burau representation of the braid groups.

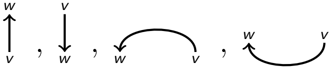
(d) Weight functions



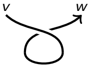
$$\xrightarrow{\mathrm{W}_s} \langle \varphi_{w_0,w_1} | r | \varphi_{v_0,v_1} \rangle = e^{w^* U v}$$



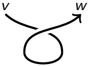
$$\xrightarrow{\mathrm{W}_s} \langle \varphi_{w_0,w_1} | r^{-1} | \varphi_{v_0,v_1} \rangle = e^{w^* U^{-1} v}$$



$$\xrightarrow{\mathrm{W}_s} e^{\bar{w} v}$$

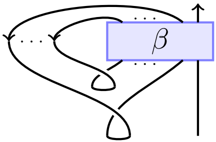
$$\mathrm{W}_{\partial s}\Bigg($$ 

$$\Bigg) = \int_{\mathbb{C}} e^{(\bar{w}\ \bar{u})\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix}} \mathrm{d}\mu_1(u) = e^{\bar{w} v}$$

$$\mathrm{W}_{\partial s}\Bigg($$ 

$$\Bigg) = \int_{\mathbb{C}} e^{(\bar{w}\ \bar{u})\begin{pmatrix} 0 & 1 \\ t^{-1} & 1-t^{-1} \end{pmatrix}\begin{pmatrix} v \\ u \end{pmatrix}} \mathrm{d}\mu_1(u) = t e^{\bar{w} v}$$

(e) Final calculation

$$D_\beta = \boxed{D_\beta} =$$ 

with the writhe $g(D_\beta) = g(\beta) + n - 1 \in 2\mathbb{Z}$ and the Burau matrix $\psi_n(\beta) = \begin{pmatrix} \hat{\beta}_n & b_\beta \\ c_\beta & d_\beta \end{pmatrix}$,

$$\mathrm{W}_{\partial s}\left(\overset{w}{\underset{v}{\boxed{D_\beta}}} \right) = \int_{\mathbb{C}^{n-1}} e^{\left(u^* \ \bar{w} \right) \psi_n(\beta) \left(\begin{smallmatrix} u \\ v \end{smallmatrix} \right)} \, \mathrm{d}\mu_{n-1}(u)$$

$$= \frac{e^{\bar{w} d_\beta v + \bar{w} c_\beta (I_{n-1} - \hat{\beta}_n)^{-1} b_\beta v}}{\det(I_{n-1} - \hat{\beta}_n)} = \frac{e^{\bar{w} v}}{\det(I_{n-1} - \hat{\beta}_n)}$$

provided $d_\beta + c_\beta (I_{n-1} - \hat{\beta}_n)^{-1} b_\beta = 1$

8: Concluding remarks

- B_1 admits a q-deformation to a quantum group $\mathrm{B}_q \simeq BU_q(sl_2)$ with $U_q(sl_2) \subset D(\mathrm{B}_q)$ so that the colored Jones polynomials are contained in the universal invariant $Z_{\mathrm{B}_q}(K)$.
- The case of roots of unity $q^N = 1$, $q^k \neq 1$, $0 < k < N$. In this case, $\mathrm{B}_1 \subset \mathrm{B}_q$ with $a \mapsto a^N$, $b \mapsto b^N$. Based on the results of Habiro, Willetts, and Brown–Dimofte–Geer, one expects that the universal invariant is of the form $Z_{\mathrm{B}_q}(K) = \frac{ADO_K(z,q)}{\Delta_K(a^N)}$ (work in progress with Shamil Shakirov).
- The case $N = 1$, seen as a limit $q \to 1$, provides a conceptual interpretation for the Melvin–Morton–Rozansky conjecture proven by Bar-Nathan and Garoufalidis.