Stability and Testability of Permutations Equations

Alex Lubotzky

Hebrew University

The starting point of our work was:

Theorem (Arzhantseva-Paunescu 2015)

For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $A, B \in Sym(n)$ satisfying $d_n([A,B], id) < \delta$, then there exists $A', B' \in Sym(n)$ s.t. $d_n(A', A),\ d_n(B', B) < \varepsilon$ and $[A', B'] = A'^{-1}B'^{-1}A'B' = id$

here:

$$d_n(\sigma, \tau) = \frac{1}{n}\#\{i \in [n] | \sigma(i) \neq \tau(i)\} \quad \text{for } \sigma, \tau, \in Sym(n).$$

Namely: if two permutations nearly commute then they are near permutations that truly commute.

This result is inspired by a long tradition in mathematical physics, where it has been studied:

Assume A, B are $n \times n$ complex matrices satisfying some property P (e.g. self adjoint/unitary) and almost commute w.r.t. some norm (e.g. Hilbert-Schmidt operator, etc.). Are they near (w.r.t. this norm) matrices (with P) which truly commute?

Many papers; the answer(s) depend on P and the norm.

One can ask such question w.r.t. any system of equations:

Let

$$(\underline{X}) = (x_1, \ldots, x_d)$$
$$R = \{r_i(\underline{X})\}_{i=1}^{k}$$

when $r_i(\underline{X}) \in F_d$ - the free group on $\underline{X}$.

Say R is stable if $\forall \varepsilon > 0,\ \exists \delta > 0$ s.t. if $(\underline{A}) = (A_1, \ldots, A_d) \in (Sym(n))^d$

and $d_n(r_i(\underline{A}),\ id) < \delta$

then $\exists (\underline{A}') = (A'_1, \ldots, A'_d) \in (Sym(n))^d$

with $d_n(A'_j, A_j) < \varepsilon,\ \forall 1 \leq j \leq d$

and $r_i((\underline{A}')) = id, \quad \forall 1 \leq i \leq k$.

Crucial observation (Glebsky-Rivera 2009, [AP])

The stability of R depends only on Γ!

i.e. if

$$\Gamma = \langle X; R\rangle \simeq \langle Y : S\rangle$$

then R is stable iff S is stable so we can define

Γ to be stable iff the relations presenting it are stable

This notion of stability can be generalized to any (finitely generated) group, not necessarily finitely presented.

Def:

Γ is stable if whenever $\varphi_n : \Gamma \to Sym(n)$ maps satisfying:

for every $g, h \in \Gamma$,

$$\lim_{n\to\infty} d_n(\varphi_n(gh), \varphi_n(g)\varphi_n(h)) = 0,$$

then there exist homomorphisms $\Psi_n : \Gamma \to Sym(n)$,

s.t. $\forall g \in \Gamma$,

$$\lim_{n\to\infty} d_n(\Psi_n(g),\ \varphi_n(g)) = 0$$

So basic question: **When is Γ stable?**

Till a few years ago only handful of results were known:

(1) Free groups are stable (trivial)

(2) [GR] finite groups are stable

(3) [AP] Abelian groups are stable

Now we know more: [AP] was very influential as it presented many open problems.

The same true for [GR] for the following observation.

Observation:

If Γ is a sofic group which is stable then Γ is residually finite.

Recall • Γ is residually finite if $\exists \Psi_n : \Gamma \to Sym(n)$ homomorphisms s.t. $\forall 1 \neq g$, $d_n(\Psi_n(g), id) = 1$ for n suff. large

- Γ is sofic if $\exists \varphi_n : \Gamma \to Sym(n)$ maps s.t.

$$\forall g, h \in \Gamma, \lim_{n\to\infty} d_n(\varphi_n(yh), \varphi_n(g)\varphi_n(h)) = 0$$

and

$$\forall 1 \neq g \in \Gamma, \lim_{n\to\infty} d_n(\varphi_n(g), id) = 1$$

Pf of observation.

If Γ a sofic there exists almost-homomorphisms as in the definition. If also stable, they can be replaced by nearby homo's Ψ_n and so Γ is residually finite. □

Corollary

If Γ is stable and **not** residually finite then it is **not** sofic.

This gives a potential method to answer the long standing open problem:

Problem (Gromov-Weiss, 80's)

Are all groups sofic?

In the last few years this philosophy was implemented in other categories, but still open for sofic & symmetric groups.

Before describing what is known here, let's point out a connection with TCS.

Connection with Property Testing (based on: Becker-Lubotzky-Mosheiff 2021)

Def: (A (q,ε)-testability) Let $A =$ finite set, $P_n \subseteq A^n$.

The **membership** of $\alpha \in P_n$ is **testable (or** P_n is (q,ε)-testable)

if $\exists 0 < \varepsilon \in \mathbb{R},\ q \in \mathbb{N}$ and a random algorithm ("**tester**")

which queries only q (independent of n) coordinates of α

and answers YES if $\alpha \in P_n$

while the answer is NO with probability $\geq \varepsilon\, dist\,(\alpha, P_n)$

- dist = normalized Hamming distance

Observation 1

Stability implies Testability of the set of solutions $P_n \in Sym(n)^d$, w.r.t. the algorithm: Given $\alpha \in Sym(n)^d$, $\alpha = (\alpha_1, \ldots, \alpha_d)$. Close random $i \in [n]$ and check if $r_j(\alpha_1, \ldots, \alpha_d)(i) = i$ for every $j = 1, \ldots, k$

Example

$R =$ the commutative relation $= (A, B)$ choose $i \in [n]$ and check if $AB(i) = BA(i)$.

If true for $q = q(\varepsilon)$ of the i's then with high probability (A, B) is near $W = \{(A', B') \in Sym(n)^2 | A'B' = B'A'\}$ by [AP]-theorem!

Observation 2

Testability of the relations R also depends only on $\Gamma = \langle X; R \rangle$ and not on R.

Program

Develop methods to decide for a group Γ whether it is testable? stable?

Summary of various results

(I) $\Gamma = \langle S \rangle$ **amenable**

(i.e. $\forall \varepsilon > 0,\ \exists F \subseteq \Gamma$ finite with $|sF \triangle F| < \varepsilon |F|,\ \ \forall s \in S$)

Theorem 1 [BLM, 2021]

Every amenable group is testable.

The proof follows from deep results of Orenstein-Weiss (1980) and Newman-Sohler (2013) ("hyperfinite")

Theorem 2 [Becker-Lubotzky-Thom, 2019]

A f.g. amenable group Γ is stable iff the finite index IRSs of Γ are dense in the space of all IRSs of Γ

Recall: An IRS (Invariant Random Subgroup) μ on Γ is a probability measure on the (compact) space $Sub(\Gamma)$ of all subgroups of Γ ($Sub(\Gamma)$ is considered as a subset of $\{0,1\}^{\Gamma}$) which is invariant under conjugation.

Ex: **(i)** Every $N \triangleleft \Gamma$ defines a Dirac measure.

(ii) μ is **finite index** IRS if its support is entirely on finite index subgroups

(iii) Prop. (Abert-Glasner-Virag 2014)
If Γ acts p.m.p. (probability measure preserving) on a probability space (Y, τ), then the stabilizer of a τ-random point is IRS.

Moreover, every IRS is obtained like that!

Cor to [BLT]

Virtually polycyclic groups are stable ([AP] proved for abelian; was not known for virt. abelian, not even abelian $\times$ finite).

Baumslag-Solitor group $BS(1,n)$ is stable.

But **not** all solvable groups are stable

Theorem 3 [BLT] The Abels group (1979); p prime

$$\left\{ \begin{pmatrix} 1 & * & * & * \\ & p^m & * & * \\ & & p^n & * \\ & & & 1 \end{pmatrix} \in GL_4(\mathbb{Z}[\frac{1}{p}]) \;\middle|\; m,n \in \mathbb{Z} \right\}$$

is not stable

Reason: It has a finitely generated normal subgroup (in fact, central) which is not closed in the profinite topology

Open Problem. Characterize the solvable stable groups!

Conjecture. Meta-abelian groups are stable!

Remark

If true it will be a significant strengthening of the classical result of P. Hall asserting the meta-abelian groups are residually finite.
While Hall's thm is proved by comm. alg. methods, the conjecture probably needs dynamic & ergodic theory.

Theorem 4 [Levit-Lubotzky 2021]

The lamp-lighter groups (and many others) are stable.

This uses works of Lindenstrauss and Weiss.

But open for the free meta-abelian.

[LL] result on the lamp-lighter group gave the first non finitely presented stable group, and parallely also:

Theorem 5 [Zheng 2021]

The Grigorchuk groups are stable.

Now we have many more:

Theorem 6 [Levit-Lubotzky, Zheng 2021]

There exist uncountably many stable groups

The examples we gave are the groups constructed by
B.H. Newmann in 1937:
Let M be an infinite subset of $\mathbb{N}$ and $G(M)$ the subgroup of $\prod_{n \in M} Sym(n)$ generated by $\tau = (\tau_n)$ and $\sigma = (\sigma_n)$ when $\tau_n = (1, 2)$ and $\sigma_n = (1, 2, \ldots, n)$. He showed they are all different. Lubotzky-Weiss showed (1993) they are amenable and now we show they all satisfy the IRS criterion
Zheng's examples are branch groups.

Now, assume Γ has Kazhdan Property (T)

i.e. every non-trivial irreducible representation of Γ is "bounded away" from the trivial representation. This implies that any two fin. dim. irr. rep. are "bounded away" from each other.

Ex: $\Gamma = SL_n(\mathbb{Z})$, $n \geq 3$ (but **not** $n = 2$)
and more generally all lattices in simple Lie groups of rank ≥ 2

Theorem 7 [Becker-Lubetzky 2020]

If Γ is a sofic group (e.g. res. finite, linear) with (T) then Γ is **not** stable.

Theorem 8 [B-L-Mosheiff 2021]

It is also not testable

Similar results with (τ) instead of (T)

Sketch of proof for $\Gamma = SL_3(\mathbb{Z})$ (á la Ozawa)

$SL_3(\mathbb{Z})$ acts, via $SL_3(\mathbb{F}_p)$, 2-transitively on $X =$ pairs of 1-dim subspaces of $\mathbb{F}_p^3$.

Thus the rep
φ on $L_0^2(x) = \{f : x \to \mathbb{C} | \varepsilon f(x) = 0\}$ is irreducible

Let $n = |X|$ and drop on point x_0 of X, to get an "almost action" on $Y = X \setminus \{x_0\}$.

If Γ is stable, then this almost action is near

true action on Y which induces a rep φ_0 on $L_0^2(Y)$ nearby φ.
This contradicts (T). $\square$

This led [BL] to define:

Def.

Γ is flexible stable if every almost action $\varphi : \Gamma \to Sym(n)$ is near a true action $\varphi : \Gamma \to Sym(N)$ with $N = n(1 + o(1))$.

Question: Is $\Gamma = SL_n(\mathbb{Z})$ flexible stable?

Remark **(1)** Γ sofic & flexible stable $\Rightarrow$ Γ res. finite

(2) Up to now we do not know any group which is flexible stable and known to be not stable.

Theorem 9 [Lazarovich-Levit-Minsky 2021]

Surface groups $T_g = \{a_1, \dots, a_g,\ b_1, \dots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1\}$ are flexible stable

Theorem 10 [Bowen-Burton 2021]

If for some $n \geq 5,\ \Gamma = SL_n(\mathbb{Z})$ is flexible stable then there exists a **non**-sofic group.

Finally:

Theorem 11 [Levit-Lazarovich 2021]

Virtually free groups are stable.

Open problem: Assume $(\Gamma : \triangle) < \infty$. Is Γ stable $\Leftrightarrow \triangle$ stable?