When can a totally disconnected locally compact group be called computable?

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Abstract: Countably infinite groups such as $\mathbb{Z}, \mathbb{Q}$, and $SL_n(\mathbb{Q})$ are computable in the sense that they have an isomorphic copy with domain the natural numbers and the group operations computable function. (Such a definition works for any countable structure in a finite signature, not only for groups.) For f.g. groups, computability in this sense is equivalent to having a decidable word problem.
A topological group G is usually uncountable. One can attempt to define computability of G via a computable structure of approximations to the elements of G. We introduce two definitions of computability for t.d.l.c. groups G and show their equivalence. One approach is to view G as embedded into $\mathrm{Sym}(\mathbb{N})$ and use certain finite injections as approximations. In another approach, one takes the ordered groupoid of compact open cosets of G as the approximation structure. Common t.d.l.c. groups such as $\mathrm{Aut}(T_d)$ and $SL_n(\mathbb{Q}_p)$ turn out to be computable in our sense.
In the abelian case, we address the question to which extent taking the Pontryagin dual preserves the computability of a t.d.l.c. group.

Background on computable structures

(first countable, then uncountable)

Computable functions on $\mathbb{N}$

- A function $f \colon \mathbb{N}^k \to \mathbb{N}$ is called computable if there is a Turing machine that on inputs $n_1, \ldots, n_k$ outputs $f(n_1, \ldots, n_k)$.
- Church-Turing thesis: computability in this sense is the same as being computable by some algorithm.

Computable structures

A structure S is given by a nonempty domain D with relations and functions defined on it.
If $D = \mathbb{N}$ and these functions and relations are computable, we say that S is a computable structure.
More generally, we say that a structure S is computable if it has a computable "copy" with domain $\mathbb{N}$.

Example

- Groups such as $\mathbb{Z}$ and $\mathbb{Q}$, and even $SL_n(\mathbb{Q})$, are computable in this sense.
- A finitely generated group is computable $\Longleftrightarrow$ its word problem is decidable.

How to define computability for uncountable structures?

Broadly speaking, one uses a computable, countable "structure" of approximations to elements.

- $(\mathbb{R}, +, \times)$ is computable as the completion of $(\mathbb{Q}, +, \times)$ w.r.t. the Euclidean metric.
- The structure $(\mathbb{Q}_p, +, \times)$ is also computable, as the completion of $(\mathbb{Q}, +, \times)$ w.r.t. the p-adic metric.

A profinite group G is computable if $G = \varprojlim(A_i, \psi_i)$ for a computable diagram $(A_i, \psi_i)_{i\in\mathbb{N}}$ of finite groups and epimorphisms ψ_i.
(La Roche 1981, Smith 1981)

A computable presentation of $\mathrm{Sym}(\mathbb{N})$

For strings σ_i, $i = 0, 1$ with natural number entries, of the same length $N \leq \infty$, $\sigma_0 \oplus \sigma_1$ denotes the string of length $2N$ which alternates between σ_0 and σ_1. E.g. $\sigma_0 = (1, 3), \sigma_1 = (4, 0)$, yields $(1, 4, 3, 0)$.
The approximation structure for $\mathrm{Sym}(\mathbb{N})$ is the computable tree $\mathrm{Tree}_{\mathrm{Sym}(\mathbb{N})} = \{\sigma \oplus \tau :$

$$\sigma, \tau \text{ are 1-1} \wedge \sigma(\tau(k)) = k \wedge \tau(\sigma(i)) = i \text{ whenever defined}\}.$$

- The paths of $\mathrm{Tree}_{\mathrm{Sym}(\mathbb{N})}$ can be viewed as the permutations of $\mathbb{N}$, paired with their inverses:
  $[\mathrm{Tree}_{\mathrm{Sym}(\mathbb{N})}] = \{f \oplus f^{-1} \colon\ f \in \mathrm{Sym}(\mathbb{N})\}$.
- For any functions f_0, f_1, g_0, g_1 on $\mathbb{N}$,
  - $(f_0 \oplus f_1)^{-1} = f_1 \oplus f_0$.
  - $(f_0 \oplus f_1) \cdot (g_0 \oplus g_1) = f_0 \circ g_0 \oplus g_1 \circ f_1)$.
- These operations of $\mathrm{Sym}(\mathbb{N})$ are given by computable functions on $\mathrm{Tree}_{\mathrm{Sym}(\mathbb{N})}$, such as $\sigma_0 \oplus \sigma_1 \mapsto \sigma_1 \oplus \sigma_0$ for the inverse.

Totally disconnected locally compact groups

- Van Dantzig's theorem from the 1930s says that each tdlc group has a compact open subgroup.
- One can use this to show that each tdlc group is topologically isomorphic to a closed subgroup of the symmetric group $\mathrm{Sym}(\mathbb{N})$.

Examples of tdlc groups:

- $G = (\mathbb{Q}_p, +)$, the p-adic numbers for a prime p.
- $G = \mathrm{Aut}(T_d)$ for $d \geq 3$. An undirected tree is a connected graph with no cycles. $\mathrm{Aut}(T_d)$ is the group of automorphism of the undirected tree T_d where each vertex has degree d (Tits, 1970). Each proper open subgroup of G is compact. Each compact subgroup of G is contained in the stabilizer of a vertex, or in the stabilizer of an edge.

Plan

We introduce two definitions of computability for tdlc groups G and show that they are equivalent. Recall that in the uncountable setting, to define computability one needs a computable countable approximation structure.

- The first definition is by viewing G as a closed subgroup of $\mathrm{Sym}(\mathbb{N})$ and use a tree of finite injections as the approximation structure. This is natural if G is given as a group of automorphisms, e.g. $\mathrm{Aut}(T_d)$.
- The second definition is via the ordered groupoid of compact open cosets of G as an approximation structure. This is natural e.g. for $\mathbb{Q}_p$, and $SL_n(\mathbb{Q}_p)$.

Definition 1 of computable tdlc groups:

via closed subgroups of $\mathrm{Sym}(\mathbb{N})$

Computably locally compact subtrees of $\mathbb{N}^*$

$\mathbb{N}^*$ denotes the tree of strings with natural number entries.
Let $T \subseteq \mathbb{N}^*$ be a computable subtree without dead ends.
Let $[T]$ be the set of paths. Note that

$$[T] \text{ is compact} \iff \text{each level of } T \text{ is finite.}$$

For $\sigma \in T$ let $[\sigma]_T = \{X \in [T] \colon \sigma \prec X\}$, the paths extending σ.

Definition (Computably locally compact trees)

We say that $[T]$ is computably locally compact (CLC) if $[T]$ is locally compact, and

- $\{\sigma \in T \colon [\sigma]_T \text{ is compact}\}$ is decidable.
- there is a computable function $h(\sigma, i)$ such that if $[\sigma]_T$ is compact and $\rho \in T$ extends σ, then $\rho(i) < h(\sigma, i)$ for each $i < |\rho|$. That is, $[\sigma]_T$ is compact in an effective way.

Computable tdlc groups (Def. 1)

Recall that $\mathrm{Sym}(\mathbb{N})$ is viewed as the set of paths of $T = \mathrm{Tree}_{\mathrm{Sym}(\mathbb{N})} = \{\sigma \oplus \tau :$

$$\sigma, \tau \text{ are 1-1} \wedge \sigma(\tau(k)) = k \wedge \tau(\sigma(i)) = i \text{ whenever defined}\}.$$

For $\eta \in T$, let $[\eta]_T$ be the set of paths on T extending η.

Definition

A closed subgroup G of $\mathrm{Sym}(\mathbb{N})$ is computable if its corresponding tree, namely $T_G = \{\eta \in T : [\eta]_T \cap G \neq \emptyset\}$ is computable.

Definition (Definition 1 of computably tdlc groups)

In case G is tdlc, we say that G computably tdlc if G is computable, and the tree T_G is computably locally compact.

Recall that $G \leq_c \mathrm{Sym}(\mathbb{N})$ is locally compact iff G has finite suborbits. Definition 1 is a computable version of this.

Note that the closed subgroups of $\mathrm{Sym}(\mathbb{N})$ are just the automorphism groups G of structures M with domain $\mathbb{N}$. To say that G is computable means that one can decide whether a finite injection on M can be extended to an automorphism. E.g., $G = \mathrm{Aut}(\mathbb{Q}, <)$ is computable because one can decide whether the injection preserves the ordering.

Example (of computably tdlc groups)

Let $d \geq 3$. The tdlc group $G = \mathrm{Aut}(T_d)$ is computably tdlc.

- We can decide whether a finite injection α on T_d can be extended to an automorphism by checking whether it preserves distances. Each $\eta \in \mathrm{Tree}_{\mathrm{Sym}(\mathbb{N})}$ corresponds to such an injection. So we can decide whether $[\eta]_{T_G} = [\eta]_{\mathrm{Tree}} \cap G \neq \emptyset$.
- $[\eta]_{T_G}$ is compact for every nonempty such string η.
  If η maps x to y, then it maps a ball $B_n(x)$ in T_d to $B_n(y)$. This yields a computable bound $h(\eta, i)$ as required. So T_G is computably locally compact.

Computable functions on computable tdlc groups

Definition

A function $s : G \to \mathbb{N}$ is computable if there is an oracle Turing machine that using a path $X \in [T_G]$ on its "oracle tape" (and the empty string as input) halts with output $s(X)$.

Intuitively, the machine can use as much of its oracle X as it likes, but it has to eventually come up with a value $s(X)$, using only that finite part of X.

We can e.g. study whether the scale function on a computable tdlc group G is computable in this sense. This is the function sending an element g to the scale of conjugation by g.
If a function $s : G \to \mathbb{N}$ is computable then s is continuous.
Willis (1994) has shown that the scale function is continous.

Definition of computable tdlc groups 2:

via approximation groupoids

Approximation groupoids

Groupoid: small category where each morphism has an inverse. Equivalently, usual group axioms but the group operation is partial.

Let G be tdlc. Define a groupoid $\mathcal{W}(G)$ that is also a lower semilattice. Domain: the open compact cosets of G, and $\emptyset$.

A, B, C denote such cosets. U, V, W denote compact open subgroups. $A \colon U \to V$ means that A is a right coset of U and a left coset of V.

If $A \colon U \to V$ and $B \colon V \to W$ then $A \cdot B \colon U \to W$.

- $(\mathcal{W}(G), \cdot)$ is a groupoid. E.g., if $A \colon U \to V$ then $A^{-1} \colon V \to U$.
- $(\mathcal{W}(G), \subseteq, \emptyset)$ is a lower semilattice with least element $\emptyset$, since the intersection of two cosets is empty, or again a coset.

$\mathcal{W}(G)$ satisfies the axioms of "inductive groupoids"; see e.g. Lawson's 1998 book Inverse semigroups.

Provenance of approximation groupoids

- The notion of approximation groupoids goes back to an idea of Tent, that appeared in a paper with Kechris and Tent in J. Symb. Logic 2018.
- The idea was further elaborated in a paper with Tent and Schlicht on the complexity of the isomorphism problem for oligomorphic groups that has just appeared in J. Math. Logic 2021. There we introduced the term "coarse group".
- Such structures are not groupoids, rather they have a ternary relation saying that $AB \subseteq C$, where A, B, C are cosets.
- We will see (e.g. Slide 22) that for the applications to computable structure theory we have in mind, it is necessary to have the groupoid and lower semilattice structure explicitly.

Computable tdlc groups (Def. 2)

Recall that the approximation groupoid of a tdlc group G is the countable structure $(\mathcal{W}(G), \cdot, \subseteq, \cap, \emptyset)$ on the compact open cosets together with $\emptyset$.

A groupoid is called computable if the relation $\{\langle x, y\rangle \colon\ x \cdot y \text{ is defined}\}$ is computable, and the operations are computable.

Definition (Definition 2 of computably tdlc groups)

Let G be a tdlc group. We say that G is computably tdlc if

- its approximation groupoid $\mathcal{W}(G)$ has a computable copy s. t.
- the function sending a pair of subgroups $U, V \in \mathcal{W}(G)$ to $|U : U \cap V|$ is computable.

$\mathbb{Q}_p$ is computable in this sense

Show that $G = (\mathbb{Q}_p, +)$ is computable tdlc in the sense of Definition 2.

- The open proper subgroups are of the form $U_r = p^r\mathbb{Z}_p$ for some $r \in \mathbb{Z}$. Note $\mathbb{Q}_p/U_r \cong C_{p^\infty}$.
- So each compact open coset has the form $C_{r,a}$, where $r \in \mathbb{Z}, a \in C_{p^\infty}$. Thus $U_r = C_{r,0}$.
- $C_{r,a} \cdot C_{s,b} = C_{r,a+b}$ if $r = s$, and undefined else.
- $C_{r,a} \subseteq C_{s,b}$ iff $r \geq s$ and $p^{r-s}b = a$.
- $C_{r,a} \cap C_{s,b} = \emptyset$ unless one is contained in the other.
- For $r \leq s$, we have $|U_r : U_s| = p^{s-r}$.

An approximation groupoid is a bit like a diagram describing a profinite group, but goes not only to closer approximations of elements (backwards), but also to less close ones (forward).

Equivalence of the two definitions

Theorem

G is computably tdlc in the sense of closed subgroups of $\mathrm{Sym}(\mathbb{N})$
$\iff G$ is computably tdlc in the sense of approximation groupoids.

Proof, "$\Rightarrow$": Suppose $G \leq_c \mathrm{Sym}(\mathbb{N})$. Recall that
$T_G = \{\eta \in \mathrm{Tree}_{\mathrm{Sym}(\mathbb{N})} \colon [\eta]_T \cap G \neq \emptyset\}$ is the tree for G.

- Each compact open subset of G is of the form $\mathcal{K}_D = \bigcup_{\eta \in D} [\eta]_{T_G}$ for finite D. Can decide whether $\mathcal{K}_D$ is compact. Encode such D by natural numbers; use as inputs. Can decide whether $\mathcal{K}_D \subseteq \mathcal{K}_E$.
- There are computable functions M, I such that
  $\mathcal{K}_{M(D,E)} = \mathcal{K}_D \cdot \mathcal{K}_E$ and $\mathcal{K}_{I(D)} = (\mathcal{K}_D)^{-1}$ for compact sets.

  Hence we can decide whether $\mathcal{K}_D$ is a subgroup (namely $\mathcal{K}_{M(D,D)} = \mathcal{K}_D$), and whether it is a coset (namely $\mathcal{K}_{M(D,I(D))}$ is a subgroup).

Theorem (recall)

G is computably tdlc in the sense of closed subgroups of $\mathrm{Sym}(\mathbb{N})$ $\iff$ G is computably tdlc in the sense of approximation groupoids.

Proof, "$\Leftarrow$":

- Recall $\mathcal{W}(G)$ is the computable approximation groupoid. Let $\widetilde{G}$ denote the group of permutations p that preserve the $\subseteq$-relation on $\mathcal{W}(G)$, and $p(A) \cdot B = p(A \cdot B)$ whenever $A \cdot B$ is defined.

  First we verify that $\Phi\colon G \cong \widetilde{G}$ where $\Phi(g)$ is the left translation action of g, i.e. $A \mapsto gA$ for $A \in \mathcal{W}(G)$.

- Then we show that $\widetilde{G}$ is computably tdlc as a closed subgroup of $\mathrm{Sym}(\mathbb{N})$. E.g., we have to decide whether a finite injection $\alpha = (\sigma \oplus \tau)$ can be extended to some $p \in \widetilde{G}$. Suppose σ sends A_i to B_i and τ sends A_i to C_i, where $i < n$ and the A_i, B_i, C_i are cosets.

  Then α can be extended iff $\bigcap_{i<n} B_i A_i^{-1} \cap A_i C_i^{-1} \neq \emptyset$ in $\mathcal{W}(G)$, which is decidable.

Computable tdlc abelian groups

Conditions for computability in the abelian case

By van Dantzig, if G is abelian tdlc, there is a compact open $K \leq G$ and $L = G/K$ is discrete. So the sequence $0 \to K \to G \to L \to 0$ is exact; G is a topological extension of L by K.

Theorem (Lupini, Melnikov and N.)

Let G be abelian tdlc group. The following are equivalent.

(1) G is computable tdlc

(2) $G \cong \varprojlim(A_i, \phi_i)$ for a computable diagram (A_i, ϕ_i) where

- the A_i are abelian discrete groups,
- $\phi_i \colon A_i \to A_{i-1}$ $(i > 0)$ are epimorphisms with finite kernels
- from i can compute a bound for the elements in the kernel.

(3) There are a computable profinite group K and a computable discrete group L such that G is an extension of L by K via a computable co-cycle $c \colon L \times L \to K$.

Pontryagin-van Kampen duality

Let G be an abelian locally compact group. The dual $\widehat{G}$ is the group of characters of G, with the compact open topology.

Pontryagin-van Kampen theorem: the natural embedding $G \to \widehat{\widehat{G}}$ mapping $g \in G$ to $\lambda\phi.\phi(g) \in \widehat{\widehat{G}}$ is a topological isomorphism. E.g the dual of $\mathbb{Z}$ is the unit circle, whose dual is $\mathbb{Z}$ again.

G	$\widehat{G}$
compact	discrete
compact connected	discrete torsion free
compact totally disconnected	discrete torsion
$0 \to K \to G \to L \to 0$	$0 \leftarrow \widehat{K} \leftarrow \widehat{G} \leftarrow \widehat{L} \leftarrow 0$

The duals of tdlc groups are thus the extensions of torsion discrete groups by profinite groups.

In what sense does duality hold computably?

Theorem (Lupini, Melnikov and N., 2021)

Suppose G is an abelian tdlc group.

If G is computable then $\widehat{G}$ is computably metrized Polish.
That is, $\widehat{G}$ is the completion of a dense computable subgroup D with a computable metric. E.g. $G = \mathbb{Z}$, $\widehat{G}$ the unit circle, D the rational unit circle.

If G is an extension of a torsion discrete group by a profinite group (i.e., $\widehat{G}$ is tdlc as well), then

$$G \text{ is computable} \iff \widehat{G} \text{ is computable.}$$

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