Non-commuting, non-generating graphs of groups

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Totally Disconnected Locally Compact Groups via Group Actions
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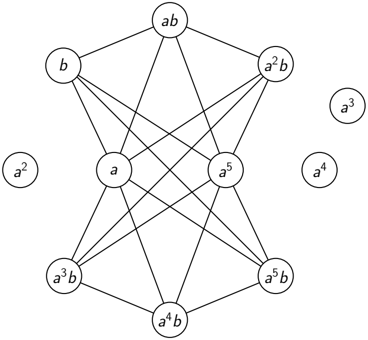

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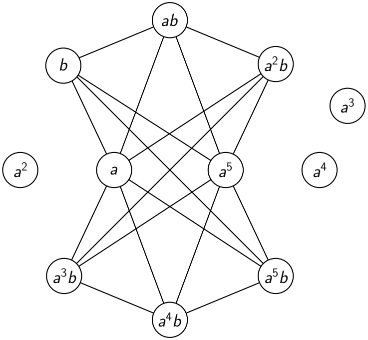

The graph is not connected, but the non-isolated vertices form a connected component of diameter 2.

A hierarchy of graphs defined on $G \setminus \{1\}$

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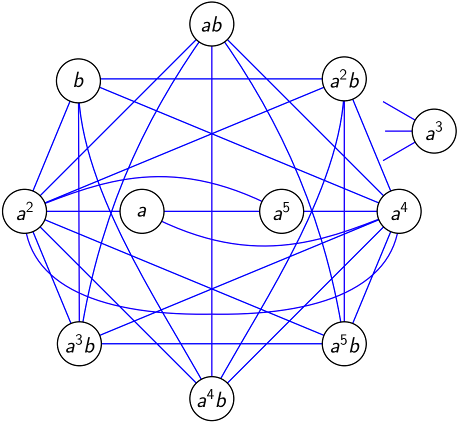

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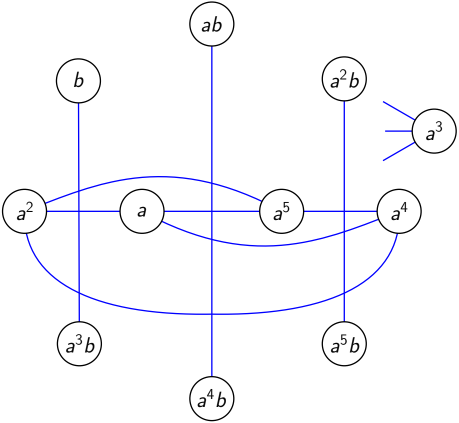

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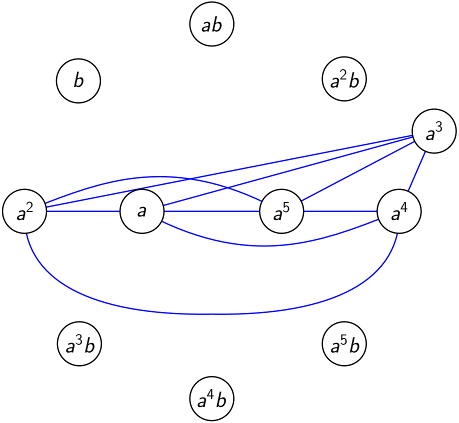

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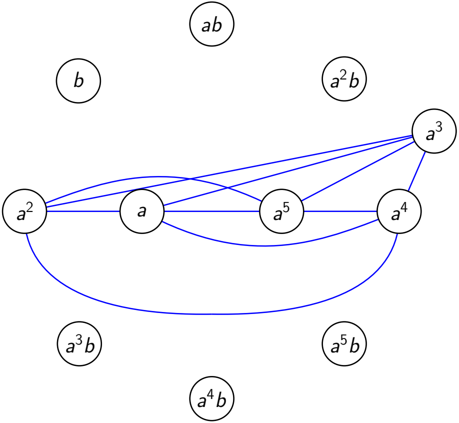

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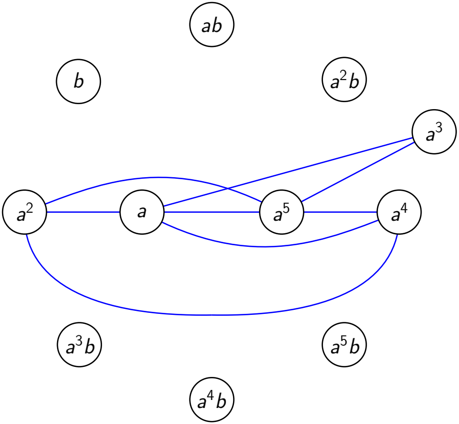

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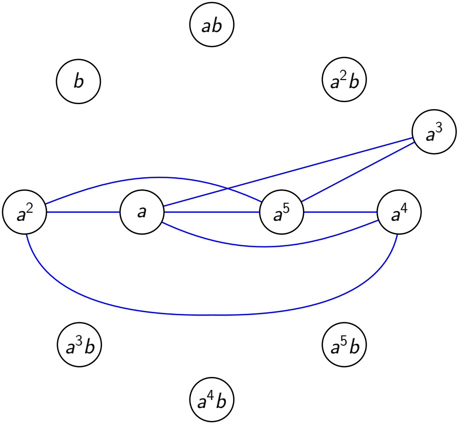

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The generating graph is the difference between the first two graphs. We will consider the next difference.

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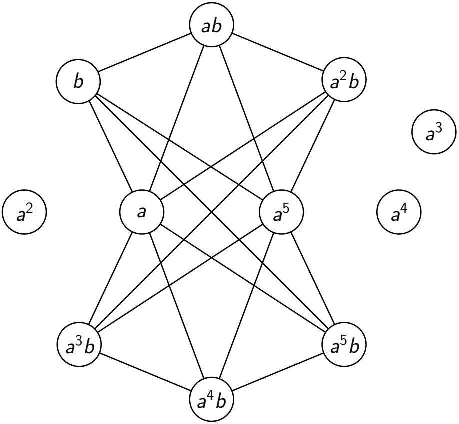

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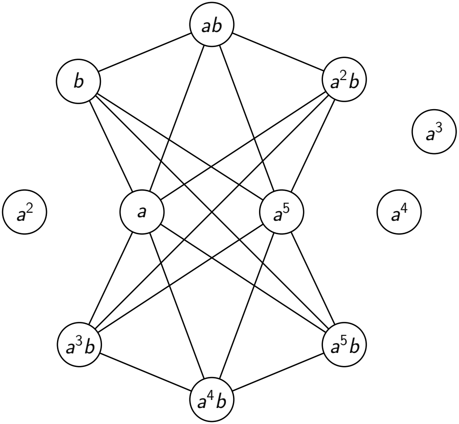

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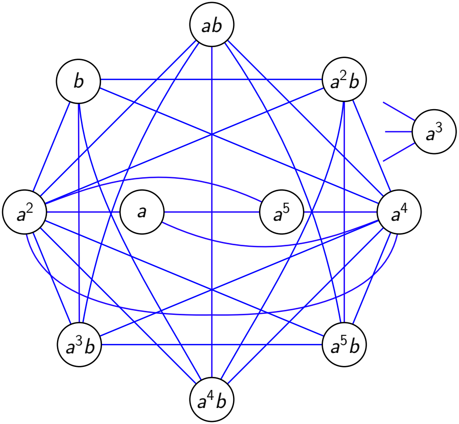

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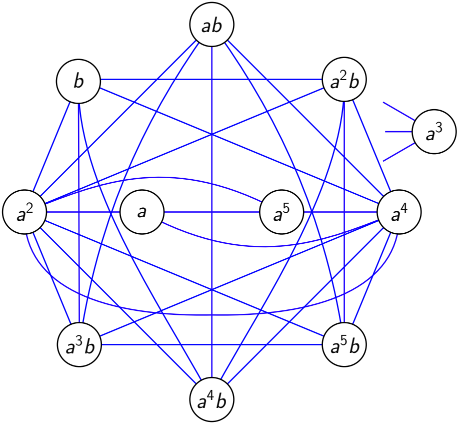

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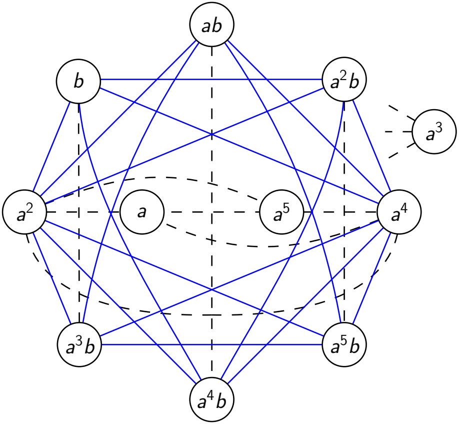

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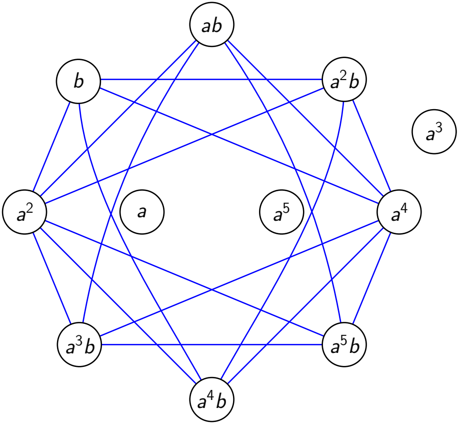

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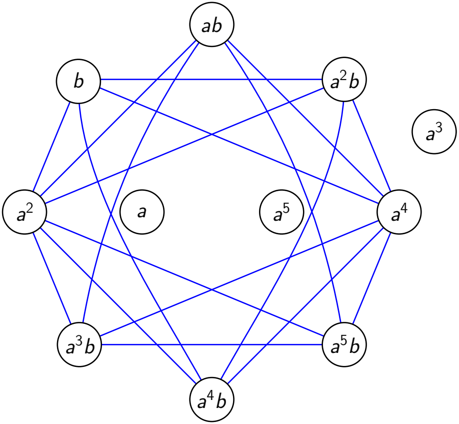

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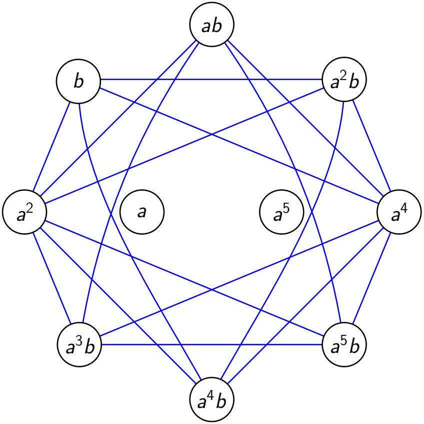

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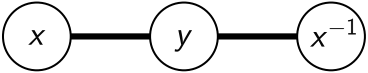

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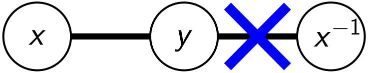

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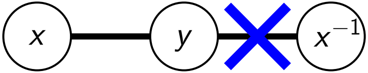

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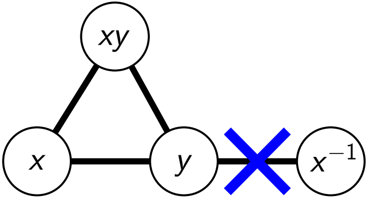

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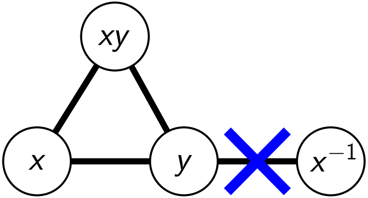

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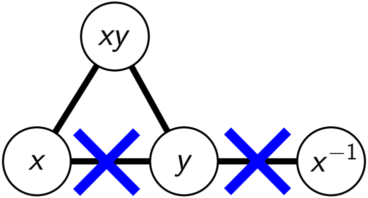

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> Ol'shanskiĭ showed in 1982 that a Tarski monster exists for each prime $p > 10^{75}$.

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If $d = 1$, then G is cyclic and hence abelian, and so $\Gamma(G)$ has no vertices.

If $d \geqslant 3$, then G has no generating pairs. Hence $\Gamma(G)$ is the non-commuting graph of G (with vertices $G \setminus Z(G)$).

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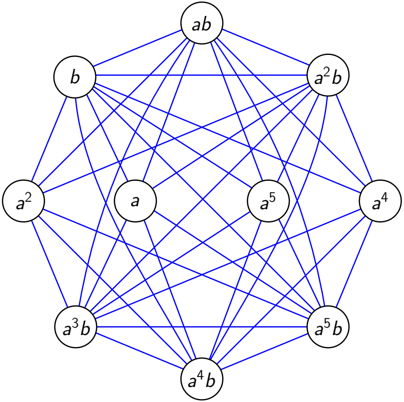

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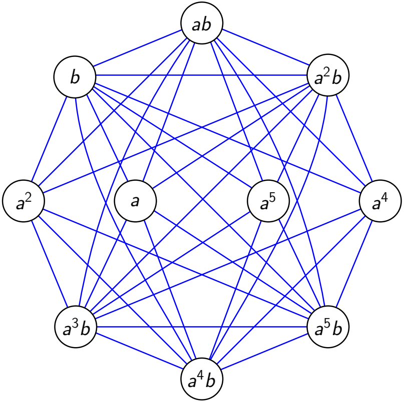

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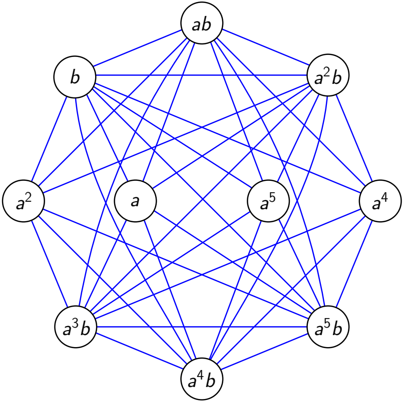

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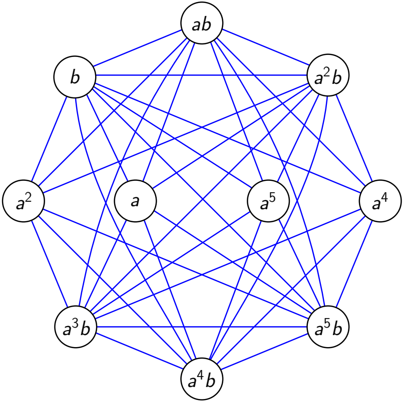

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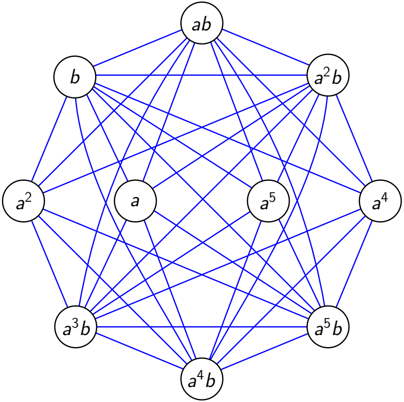

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We are therefore only interested in $\Gamma(G)$ when G is 2-generated and non-abelian.

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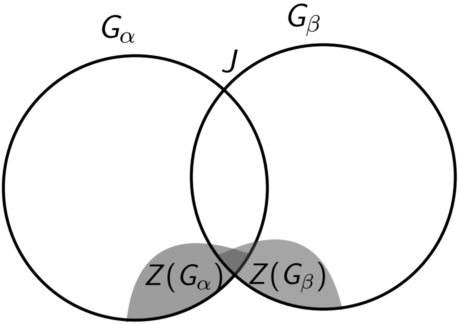

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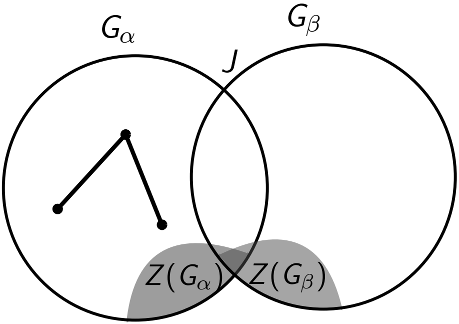

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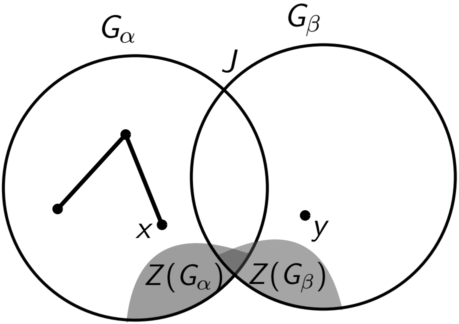

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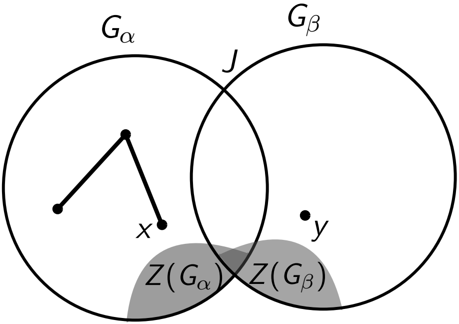

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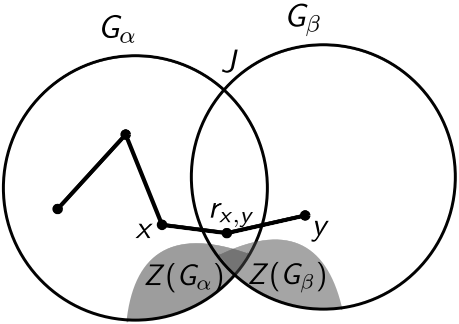

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So there exists $r_{x,y} \in J$ with $(x, r_{x,y}, y)$ a path in $\Gamma(G)$.

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Let $G := A_n$, $n \geqslant 5$. Then $\Gamma(G)$ is connected with diameter at most 4 if n is odd, and at most 3 if n is even.

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$sx \neq xs$ and $\langle s, x \rangle \leqslant N_G(\langle s \rangle) < G \implies s \sim x$.

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We'll revisit this question later.

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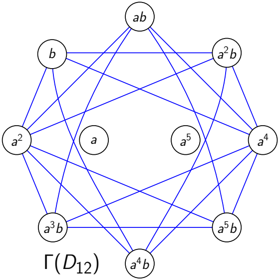

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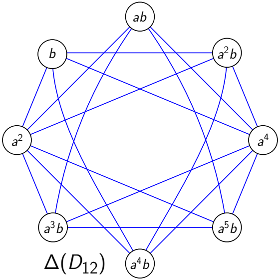

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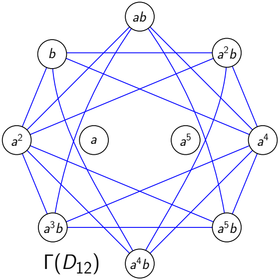

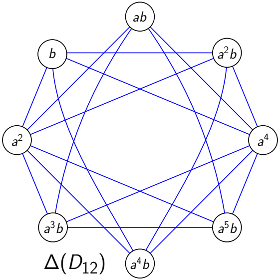

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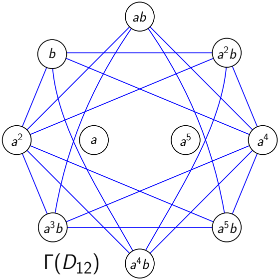

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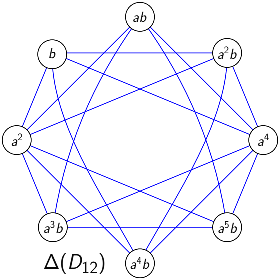

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For a finite nilpotent group G, we can prove a more precise relationship between the structures of G and $\Gamma(G)$. We use the fact that G is the direct product of its Sylow subgroups.

Direct products of groups

Lemma (Cameron, F. & Roney-Dougal, 2021)

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Example:

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Theorem (Crestani & Lucchini, 2013)

Let k be a positive integer. There exists an odd prime p and a positive integer n such that, excluding isolated vertices, the generating graph of $(\mathrm{PSL}(2, 2^p))^n$ is connected with diameter greater than k.

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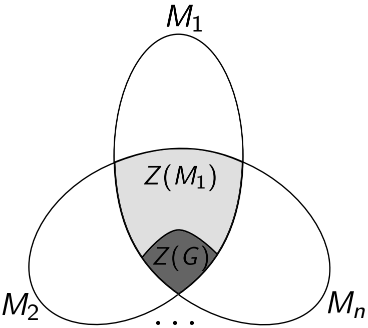
Let G be a 2-generated finite soluble group. Excluding isolated vertices, the generating graph of G is connected with diameter 2 or 3.



What about $\Gamma(G)$?

There exist 2-generated finite soluble groups G with maximal subgroups $M_1, \dots, M_n$, where for all distinct i, j: $M_i \cap M_j = Z(M_1) > Z(G)$. For $i \neq 1$, $Z(M_i) = Z(G)$.

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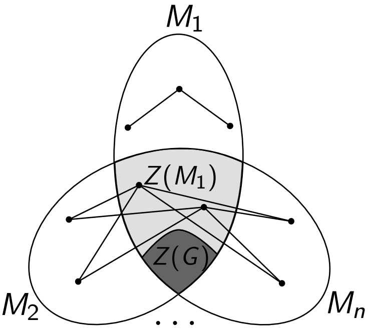

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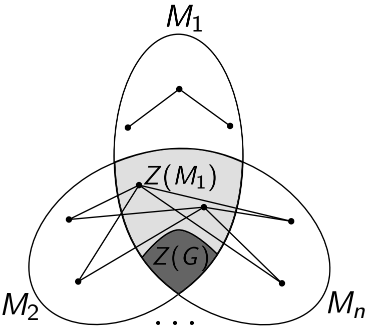

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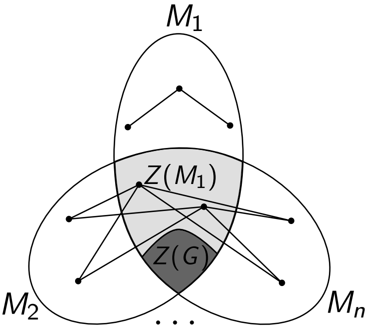

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Theorem (F., 2021+)

Let G be a finite soluble group. If G is not a $[2, 2]$-group, then $\Delta(G)$ is either empty or connected with diameter 2 or 3. If $\Delta(G)$ is connected with diameter 3, then $\Delta(G) = \Gamma(G)$.

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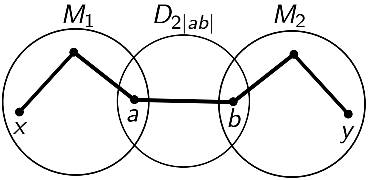

$|M_1|$ and $|M_2|$ even, $|a| = |b| = 2$

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Let H be a central extension of G. If $Z(G) = 1$ and $\Gamma(G)$ is connected with diameter k, then $\Gamma(H)$ is connected with diameter at most k.

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Question: Is there a finite insoluble group G with $\Delta(G) \neq \Gamma(G)$?

Isolated vertices, revisited

Suppose that G is non-abelian and 2-generated. A vertex x of $\Gamma(G)$ is isolated if and only if:

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Using results of Guralnick & Tracey (2021+):
G finite and simple, x satisfies (i) $\implies x \notin Z(M)$. So $\Delta(G) = \Gamma(G)$.

Finite simple groups

G	$\mathrm{diam}(\Gamma(G))$
$\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{J}_2$	2
$\mathrm{M}_{23}, \mathrm{J}_1$	3
$\mathbb{B}, \mathrm{PSU}(7, 2)$	4
Remaining sporadic groups (and ${}^2F_4(2)'$)	$\leqslant 4$
A_n; n even	$\leqslant 3$
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Question: Can these upper bounds be reduced?

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More generally, if $G = \langle a, b \mid a^r = b^s = 1 \rangle$, with $2 \leqslant r, s \leqslant \infty$, then either $G = D_\infty$ or $\Gamma(G)$ is connected with diameter 2.