2-closed groups and automorphism groups of digraphs

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k-closure of a permutation group

Wielandt(1969)

Ω a set, $G \leqslant \mathrm{Sym}(\Omega)$.

Then for each integer $k \geq 1$, we have that G acts on Ω^k via

$$(\omega_1, \omega_2, \ldots, \omega_k)^g = (\omega_1^g, \omega_2^g, \ldots, \omega_k^g).$$

The k-closure $G^{(k)}$ of G is the largest subgroup of $\mathrm{Sym}(\Omega)$ with the same set of orbits on Ω^k as G.

- If G has orbits $\Omega_1, \Omega_2, \ldots, \Omega_t$ on Ω then

$$G^{(1)} = \mathrm{Sym}(\Omega_1) \times \mathrm{Sym}(\Omega_2) \times \cdots \times \mathrm{Sym}(\Omega_t).$$

2-closure

Let $\Delta_1, \Delta_2, \ldots, \Delta_t$ be the orbits of G on Ω^2 (the orbitals of G).

The orbital digraphs of G are the digraphs Γ_i with vertex set Ω and arc set Δ_i.

We have $G \leqslant \mathrm{Aut}(\Gamma_i)$ for each i.

Note that Γ_i is a graph if and only if Δ_i is self-paired, that is, $(\alpha, \beta) \in \Delta_i$ if and only if $(\beta, \alpha) \in \Delta_i$.

- $G^{(2)} = \bigcap_{i=1}^{t} \mathrm{Aut}(\Gamma_i)$
- G is 2-transitive on Ω if and only if $G^{(2)} = \mathrm{Sym}(\Omega)$.

k-closed permutation groups

$$\mathrm{Sym}(\Omega) \geqslant G^{(1)} \geqslant G^{(2)} \geqslant G^{(3)} \geqslant \cdots \geqslant G$$

We say that G is k-closed if $G = G^{(k)}$.

If Γ is a graph or digraph then $\mathrm{Aut}(\Gamma)$ is 2-closed.

Question

Which 2-closed groups are not the automorphism group of a graph or digraph?

Bases

Wielandt: If $k \geqslant 2$ and there exists $\alpha_1, \ldots, \alpha_{k-1} \in \Omega$ such that $G_{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}} = 1$ then G is k-closed.

- If $|\Omega| = n$ then G is n-closed.
- Any semiregular permutation group[1] is 2-closed.

[1] acts freely

Cayley graphs and digraphs

Let Γ be a Cayley graph or digraph for a group G. Then G acts regularly[2] as a group of automorphisms of Γ.

If Γ is a graph we say that Γ is a GRR for G if $\mathrm{Aut}(\Gamma) \cong G$.

If Γ is a digraph we say that Γ is a DRR for G is $\mathrm{Aut}(\Gamma) \cong G$.

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[2] acts transitively and freely

GRRs

Hetzel (1976), Godsil (1981): The only groups without a GRR are

- abelian groups of exponent greater than 2.
- generalised dicyclic groups.
- C_2^2, C_2^3, C_2^4, S_3, D_8, D_{10}, A_4, $Q_8 \times C_3$, $Q_8 \times C_4$.
- 4 other groups

These are 2-closed groups that are not the automorphism group of a graph.

DRRs

Babai (1980): The only groups without a DRR are C_2^2, C_2^3, C_2^4, C_3^2 and Q_8.

These are 2-closed groups that are not the automorphism group of a digraph.

Question

Are there any 2-closed groups that are not regular and not the automorphism group of a graph or digraph?

Small rank

The rank of $G \leqslant \mathrm{Sym}(\Omega)$ is the number of orbits that it has on Ω^2.

This is also the number of orbits of G_ω on Ω.

Rank of G is at least 2 as $\Delta_0 = \{(\omega, \omega) \mid \omega \in \Omega\}$ is G-invariant.

G has rank 2 if and only if G is 2-transitive.

The only 2-closed group of rank 2 on n points is S_n, this is the automorphism group of K_n.

Rank 3

Let G be 2-closed and rank 3 with orbitals Δ_0, Δ_1 and Δ_2.

Then $\mathrm{Aut}(\Gamma_1)$ fixes Δ_1 (the set of arcs) and Δ_2 (the set of non-arcs).

Thus $\mathrm{Aut}(\Gamma)_1$ has the same orbitals as G and so $G = \mathrm{Aut}(\Gamma_1)$.

Rank 4

Theorem (Giudici-Morgan-Zhou): Let G be a finite 2-closed primitive permutation group of rank 4 that is not the automorphism group of a graph or digraph. Then one of the following holds:

- G lies in one of 2 infinite families.
- G is one of 7 groups of degree 25, 64, 81, 81, 169, 625, 2401.
- $G \leqslant \mathrm{A\Gamma L}(1, p^d)$.

Hamming graphs

$H(2,n)$ is the graph with vertex set Δ^2 where $|\Delta| = n$ and two vertices are adjacent if they differ in precisely one coordinate.

Automorphism group is $S_n \wr S_2$.

Lemma: Let $G_0 \leqslant \mathrm{GL}(V)$ preserve the decomposition $V = V_1 \oplus V_2$ such that $\dim(V_1) = \dim(V_2)$ and $B = (V_1 \cup V_2)\backslash\{0\}$ is an orbit of G_0. Then the orbital graph for $G = V \rtimes G_0$ arising from B is isomorphic to $H(2, |V_1|)$.

Since $V = V_1 \oplus V_2$, elements can be identified with (v_1, v_2), $v_1 \in V_1$, $v_2 \in V_2$.

Family 1

$V = X \otimes Y$ where X, Y have dimensions 2 and m over $\mathrm{GF}(3)$.

Let $G = V \rtimes (D_8 \circ \mathrm{GL}(m,3))$ where D_8 fixes the decomposition $X = \langle x_1 \rangle \oplus \langle x_2 \rangle$.

Now $V = V_1 \oplus V_2$ where $V_1 = \langle x_1 \rangle \otimes Y$ and $V_2 = \langle x_2 \rangle \otimes Y$.

Also $V = W_1 \oplus W_2$ where $W_1 = \langle x_1 + x_2 \rangle \otimes Y$ and $W_2 = \langle x_1 - x_2 \rangle \otimes Y$.

G_0 which has orbits

$$\{0\}, \quad (V_1 \cup V_2)\backslash\{0\}, \quad (W_1 \cup W_2)\backslash\{0\}, \quad V\backslash(B_1 \cup B_2 \cup \{0\})$$

$\Gamma_1 \cong \Gamma_2 \cong H(2, 3^m)$.

$\mathrm{GL}(2,3) \circ \mathrm{GL}(m,2) \leqslant \mathrm{Aut}(\Gamma_3)_0$.

Family 2

$V = X \otimes Y$ where X, Y have dimensions 2 and m over $\mathrm{GF}(4)$.

Let $C_3 \,\mathrm{wr}\, S_2 \leqslant \mathrm{GL}(2,4)$ preserve $X = \langle x_1 \rangle \oplus \langle x_2 \rangle$.

Let $G_0 = ((C_3 \,\mathrm{wr}\, S_2) \circ \mathrm{GL}(m,4)).2 \leqslant \Gamma\mathrm{L}(m,4)$ and $G = V \rtimes G_0$.

Again $V = V_1 \oplus V_2$ with $V_1 = \langle x_1 \rangle \otimes Y$ and $V_2 = \langle x_2 \rangle \otimes Y$.

Orbits of G_0 are $B_1 = (V_1 \cup V_2) \backslash \{0\}$,

$B_2 = \langle x_1 + x_2 \rangle \otimes Y \cup \langle x_1 + \lambda x_2 \rangle \otimes Y \cup \langle x_1 + \lambda^2 x_2 \rangle \otimes Y) \setminus \{0\}$

and $V \backslash (B_1 \cup B_2 \cup \{0\})$.

$\Gamma_1 = H(2, 4^m)$

$(\mathrm{GL}(2,4) \circ \mathrm{GL}(m,4)).2 \leqslant \mathrm{Aut}(\Gamma_3)_0$

Family 2

Y is also $2m$-dimensional over $\mathrm{GF}(2)$.

Let $u_1 = x_1 + x_2$ and $u_2 = \lambda x_1 + \lambda^2 x_2$.

Then $V = \langle u_1, u_2 \rangle_{\mathrm{GF}(2)} \otimes_{\mathrm{GF}(2)} Y$ and the simple tensors are the elements of B_2.

Thus $\mathrm{GL}(2,2) \circ \mathrm{GL}(2m,2) \leqslant \mathrm{Aut}(\Gamma_2)_0$.

Proof set up

G a 2-closed, rank 4 primitive group with orbitals $\Delta_0, \Delta_1, \Delta_2, \Delta_3$.

Γ_i is the graph with arc-set Δ_i, for $i = 1, 2, 3$.

$n \geqslant 4095$.

$\mathrm{Aut}(\Gamma_i)$ is a rank 3 primitive group.

Rank 4 primitive groups

Cuypers (1989)

If G is a finite primitive rank 4 group on Ω then one of the following holds:

1. G is affine.
2. G is almost simple.
3. $\mathrm{PSL}_2(8)^2 \lhd G \leqslant G_0 \,\mathrm{wr}\, S_2$ with $\Omega = \Delta^2$ where $|\Delta| = 28$.
4. $T^3 \lhd G \leqslant G_0 \wr S_3$, and $\Omega = \Delta^3$, and G_0 is a 2-transitive group on Δ with socle T
5. $\mathrm{soc}(G) = A_5 \times A_5$, and $\mathrm{soc}(G)_\omega = \{(t,t) \mid t \in A_5\}$.

Reduction

Cases 3, 4 and 5 are easily eliminated.

To eliminate Case 2, use

- Liebeck–Praeger–Saxl's classification of primitive groups with a common orbital and distinct socles, and
- Bamberg–Giudici–Liebeck–Praeger–Saxl's classification of the almost simple $\frac{3}{2}$-transitive groups.

Now G is an affine rank 4 group and for each $i = 1, 2, 3$ we have that $\mathrm{Aut}(\Gamma_i)$ is a rank 3 primitive group.

Again, G and $\mathrm{Aut}(\Gamma_i)$ have a common orbital (namely Δ_i).

Two cases:

- $\mathrm{soc}(\mathrm{Aut}(\Gamma_i)) = \mathrm{soc}(G)$ for all i.
- $\mathrm{soc}(\mathrm{Aut}(\Gamma_i)) \neq \mathrm{soc}(G)$ for some i.

In first case $G \leqslant \mathrm{A\Gamma L}(1, p^d)$.

In second case, use LPS again to eliminate the possibility that $\mathrm{Aut}(\Gamma_i)$ is almost simple.

Then G_0 preserves $V = V_1 \oplus V_2$ and $\Gamma_i = H(2, |V_1|)$.

Do alot of comparing of the subdegrees of rank 3 primitive affine groups from Liebeck to deduce that G is in one of our two families.

Small examples

- PrimitiveGroup(25,11). Here $G = 5^2 : (D_8.2)$. All three nontrivial orbital graphs of G are isomorphic to $H(2,5)$.
- PrimitiveGroup(64,27). Here $G = 2^6 : (3_+^{1+2} : D_8)$ has two orbital graphs of valency 18 and one of valency 27. The first two have automorphism group $2^6 : 3.A_6.2$. The orbital graph of valency 27 has automorphism group $2^6.\mathrm{GO}^-(6,2)$.
- PrimitiveGroup(81,77). Here $G = 3^4 : (2 \times Q_8) : A_4$ and has two orbital graphs of valency 16 and one of valency 48. The first two are Hamming graphs $H(2,9)$, while the latter has automorphism group $3^4 : 2^{1+4}.\mathrm{GO}^+(4,2)$.
- PrimitiveGroup(81,87). Here $G = 3^4 : (\mathrm{GL}(1,3) \wr D_8).2$. Here G has one orbital graph of valency 16 and two of valency 32. The first is isomorphic to $H(2,9)$ while the automorphism groups of the other two are isomorphic to $3^4 : 2^{1+4}.\mathrm{GO}^+(4,2)$.

Small examples II

- PrimitiveGroup(169,41). Here $G = 13^2 : (3 \times 3 : 8)$ which has an orbital graph of valency 24 (the Hamming graph $H(2, 13)$) and two of valency 72. The latter both have automorphism group $13^2 : 12 \circ 2^{1+2} : S_3$.
- PrimitiveGroup(625,547). Here $G = 5^4 : 4.A_6$ which has an orbital graph of valency 144 and two of valency 240. The graph of valency 144 has $5^4 : 4.A_6.2$ as its automorphism group. The two orbital graphs of valency 240 have $5^4 : 4 \circ 2^{1+4}\,\mathrm{Sp}(4, 2)$.
- PrimitiveGroup(2401,991). Here $G = 7^4 : C_6 \circ 2^{1+4}\Omega^-(4, 2)$ which has two orbital graphs of valency 240 and one of valency 1920. The first two have automorphism group $7^4 : C_6 \circ \mathrm{Sp}(4, 3)$ while the last has automorphism group $7^4 : C_6 \circ 2^{1+4}\mathrm{GO}^-(4, 2)$.