A local-to-global complement to Bass-Serre Theory

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(All work joint with Colin Reid)

University of Lincoln

BIRS: tdlc groups via group actions
16th August 2021

Overview

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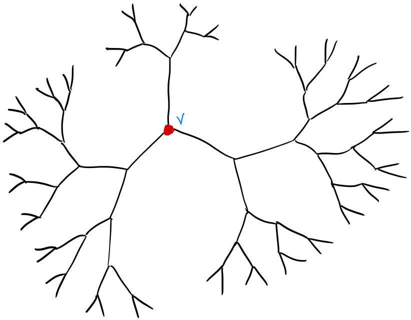

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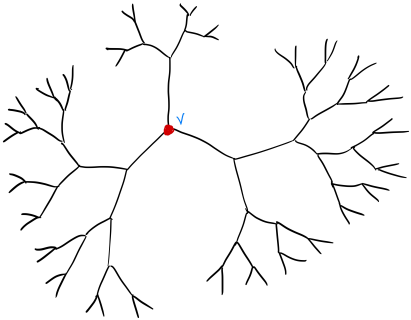

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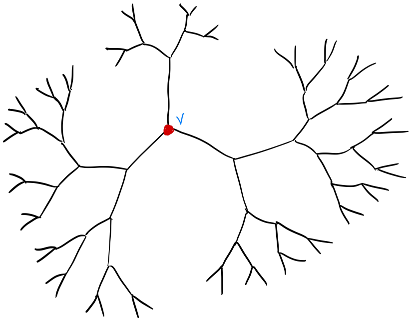

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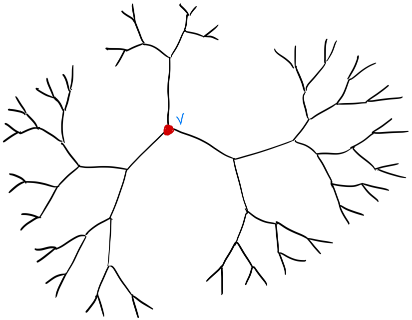

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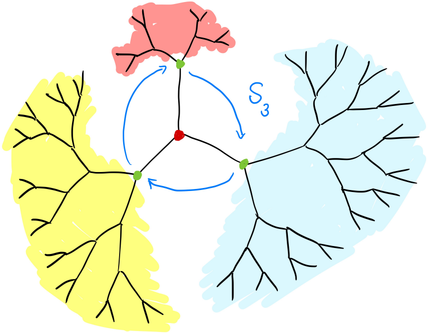

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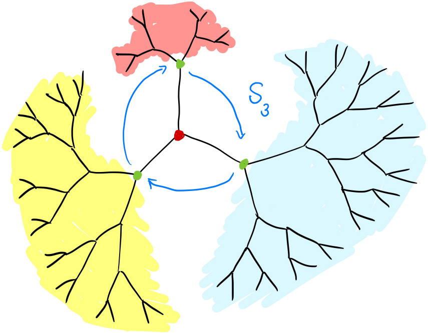

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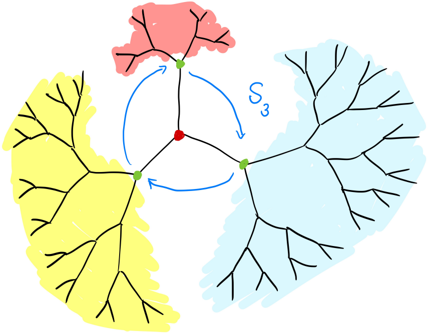

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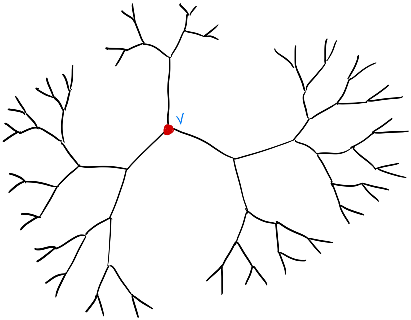

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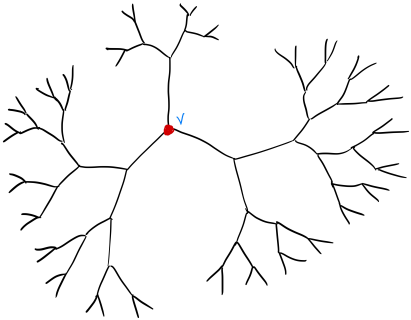

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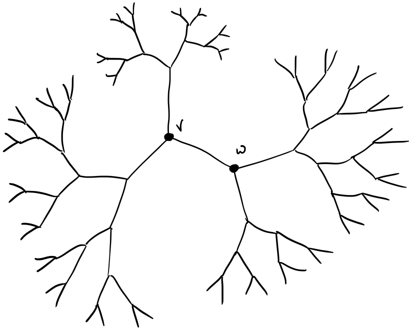

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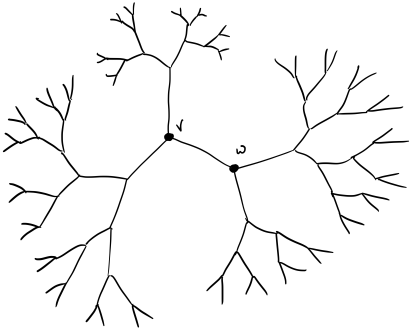

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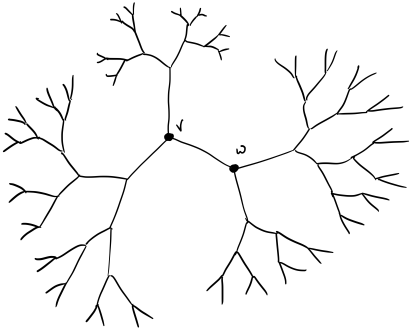

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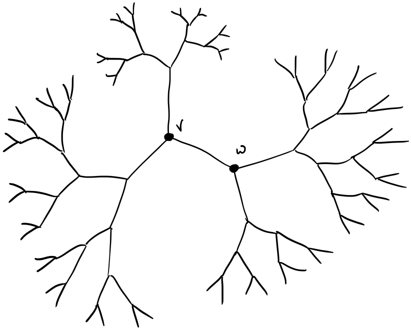

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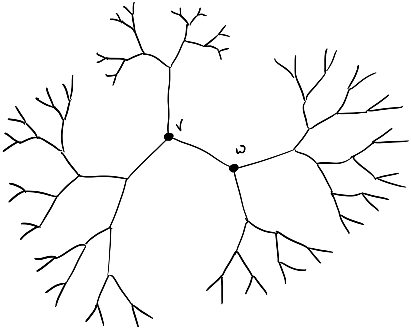

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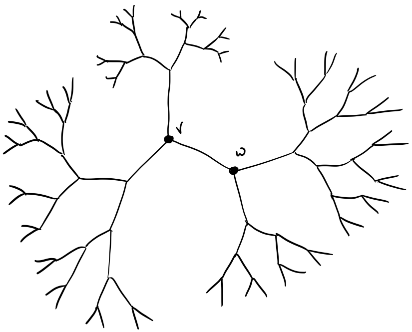

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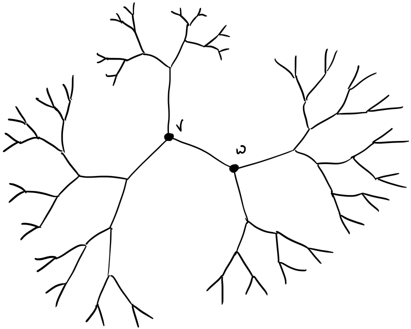

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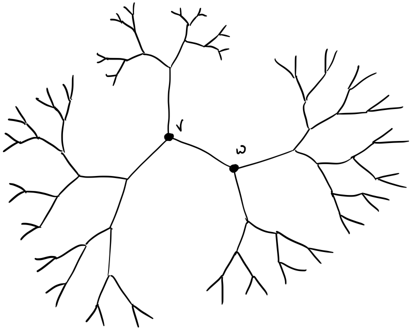
Moral: choice of local action can severely restrict global behaviour



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Groups acting on infinite trees

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◎ Construction — problems arise if you want to specify the action . . .

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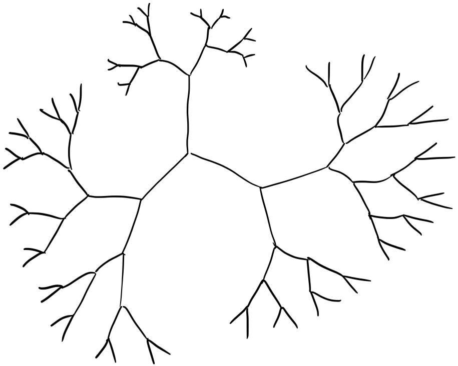

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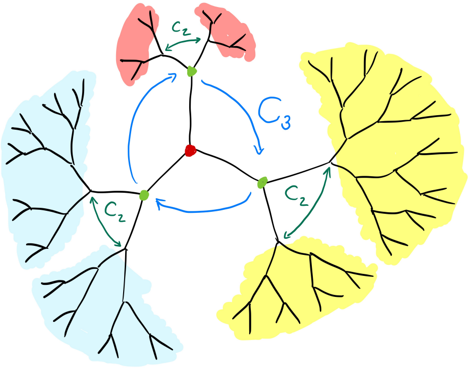

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Local-to-global constructions avoid this. We have a local-to-global complement to Bass–Serre theory

Locally compact groups

Recall a fundamental class of tdlc groups:

$\mathscr{S}$ is the class of groups that are:

- nondiscrete
- locally compact
- compactly generated
- topologically simple
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Groups acting on trees are the main source of examples of nonlinear groups in $\mathscr{S}$

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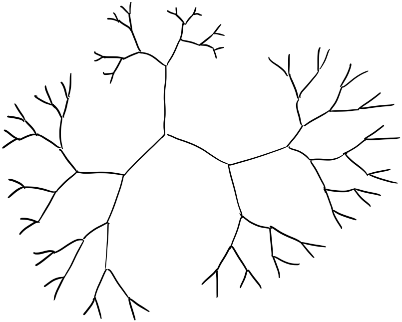

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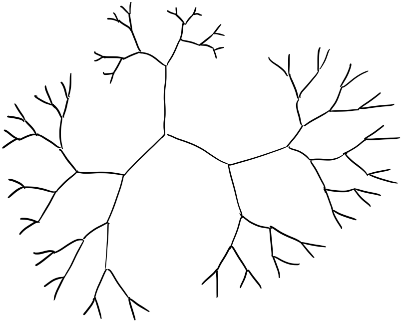

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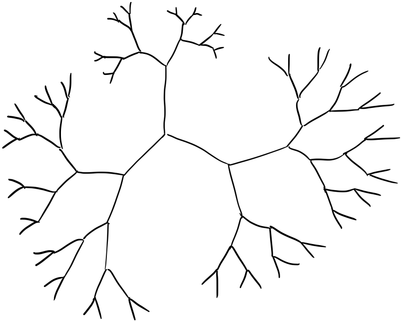

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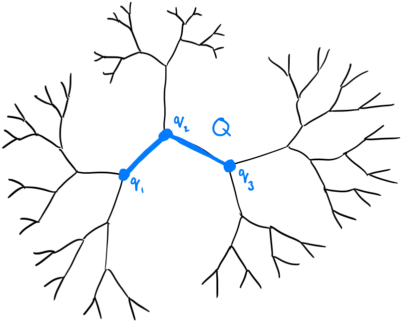

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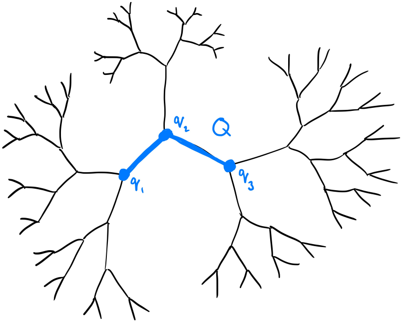

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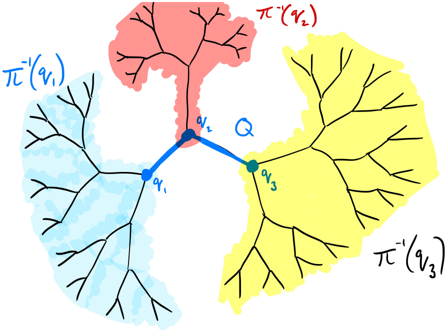

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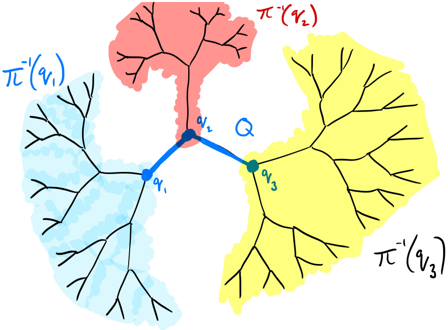

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Breakthrough:

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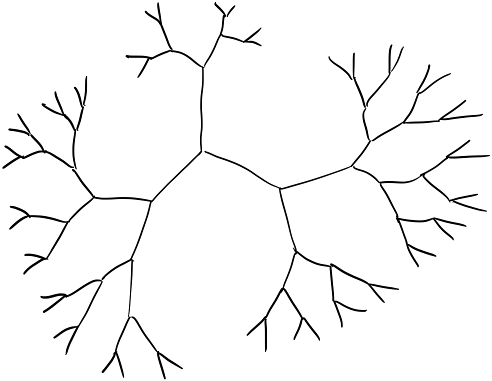

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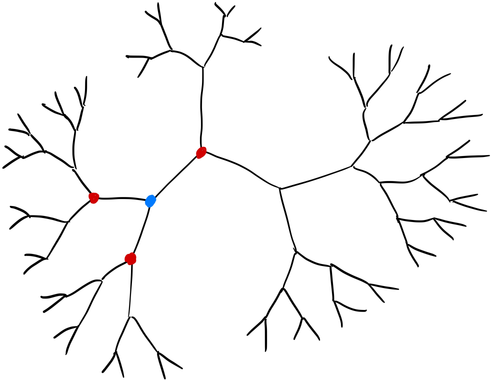

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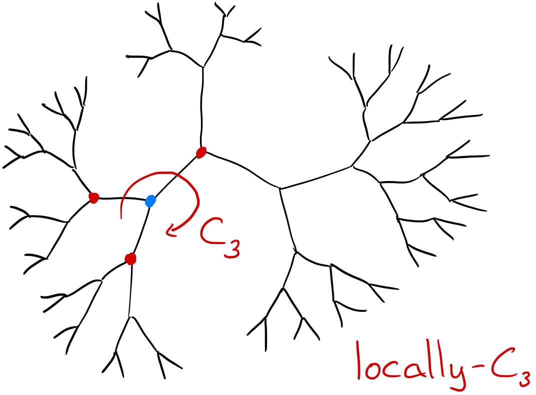

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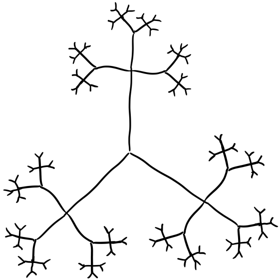

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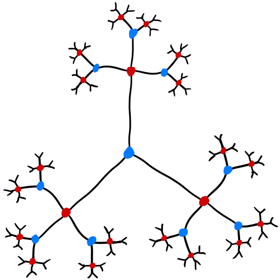

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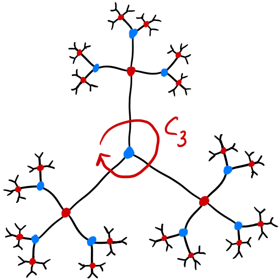

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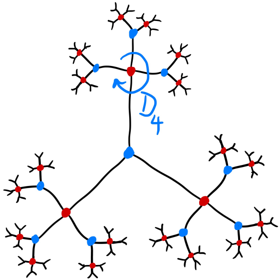

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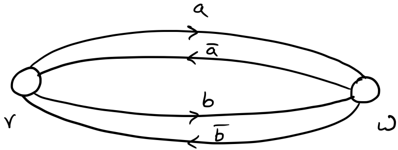

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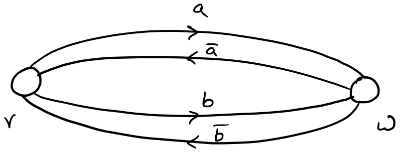

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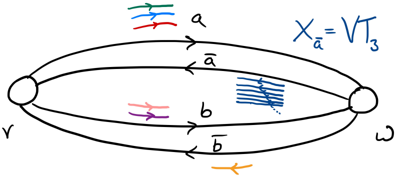

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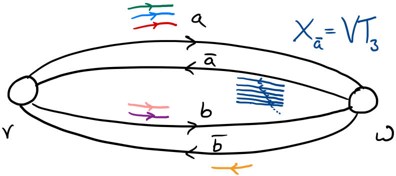

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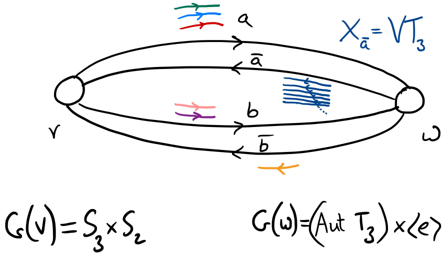

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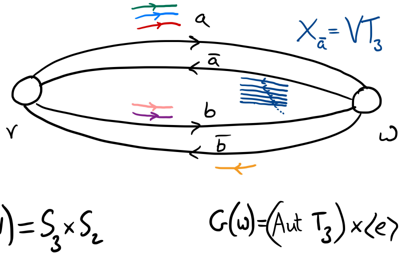

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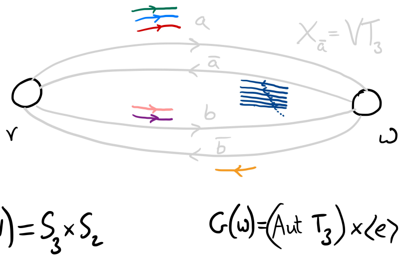

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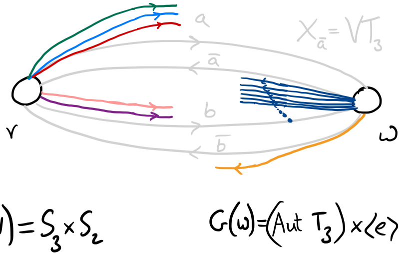

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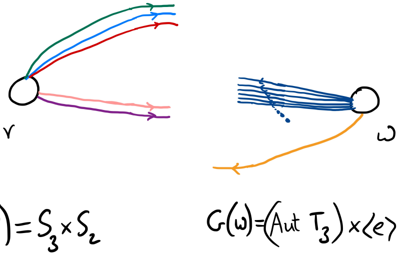

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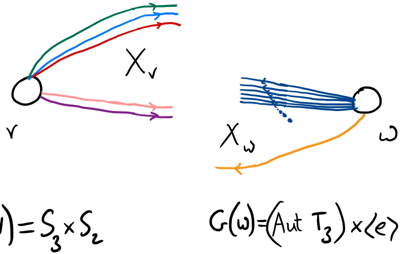

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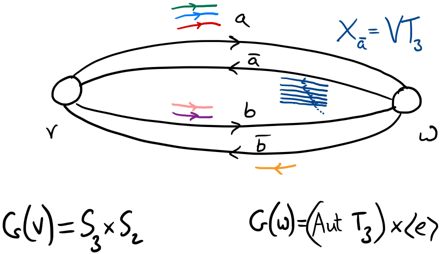

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- Gives a general way of completely describing and classifying closed subgroups of $\mathrm{Aut}\, T$ with Property (P) (call such groups (P)-closed)

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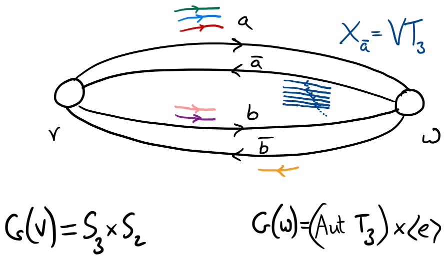

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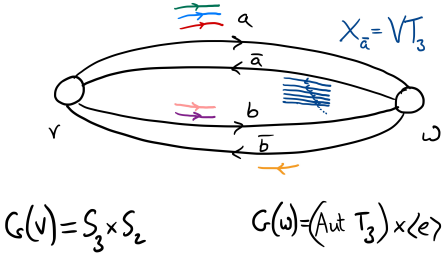

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Isomorphisms of local action diagrams: the graphs are isomorphic (with iso. θ) & the local actions are perm. isomorphic via $X_a \mapsto X_{\theta(a)}$ around each vertex

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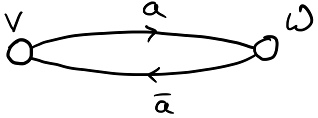

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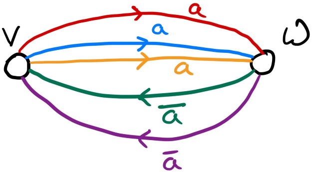

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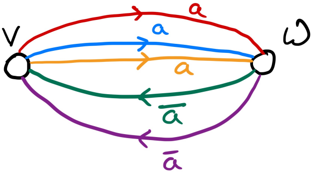

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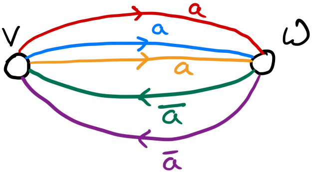

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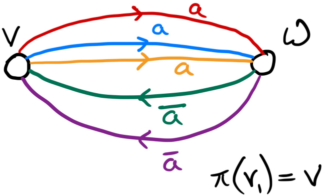

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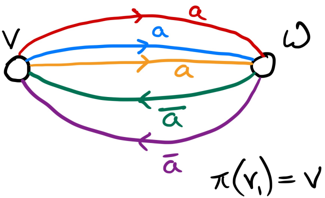

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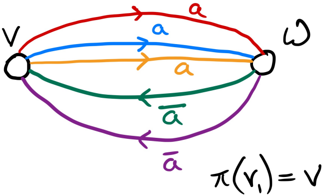

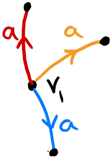

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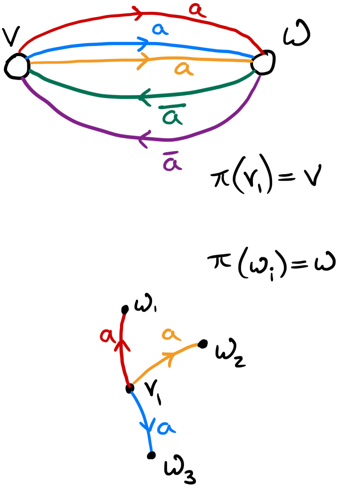

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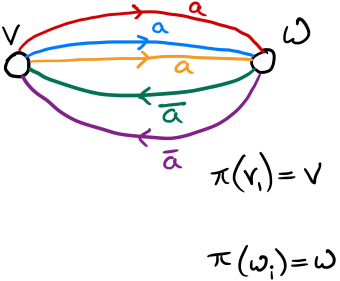

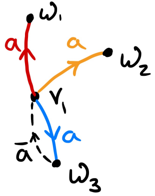

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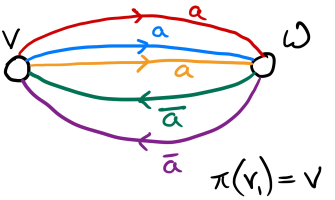

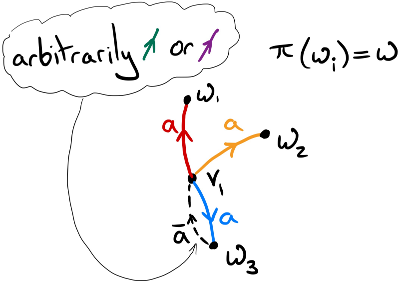

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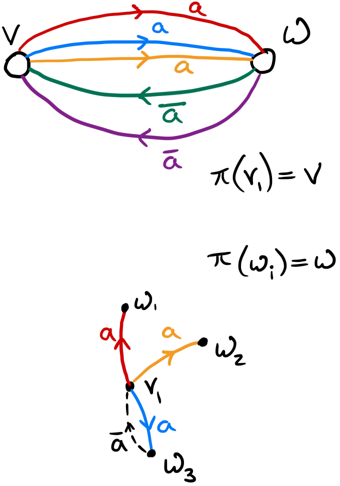

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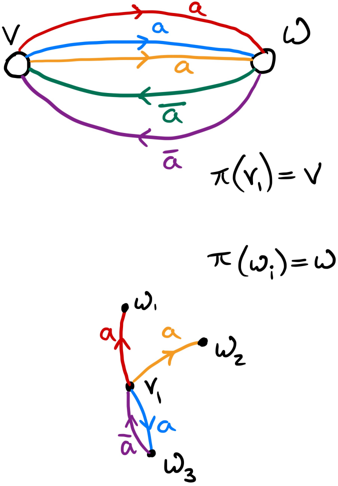

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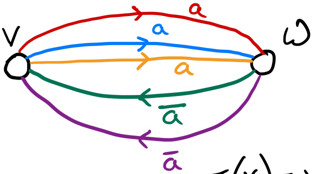

$\pi(v_1) = v$

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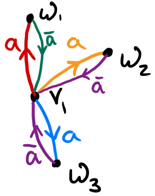

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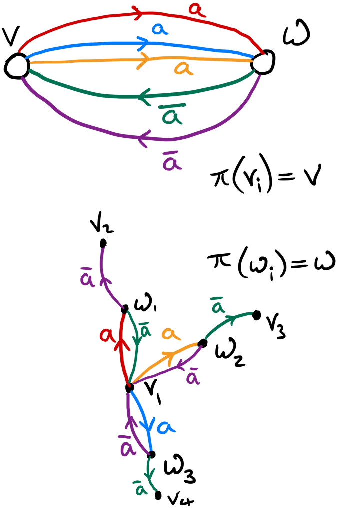

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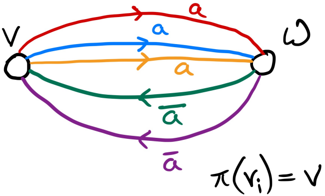

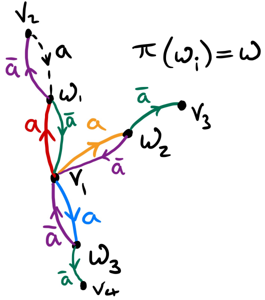

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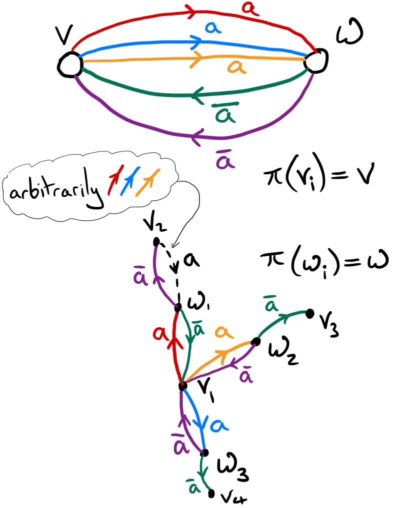

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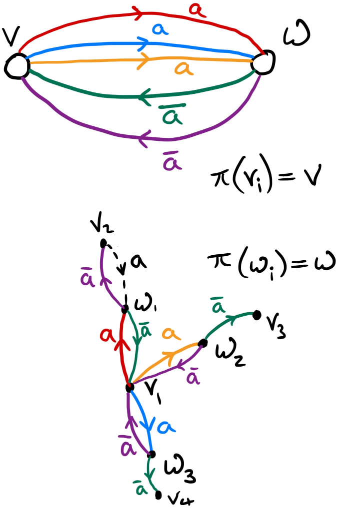

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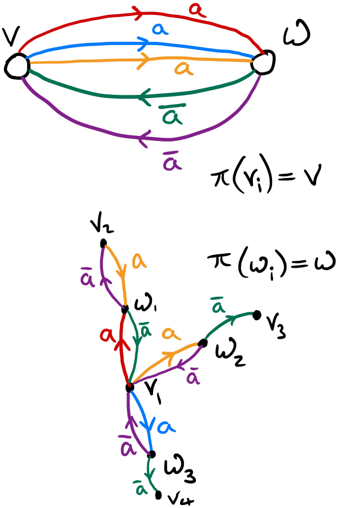

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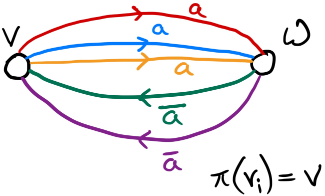

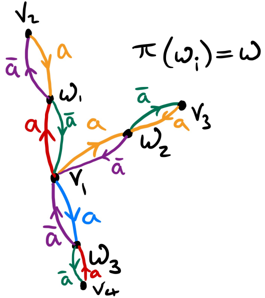

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1. Every 𝔩𝔞𝔡 Δ gives rise to a special arc-coloured tree $\mathbf{T}$ called a Δ-tree

- Δ is a graph Γ with arc-colours X_a (local actions play no part here)
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  - A colouring map
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  such that

  $\forall v_i \in VT$ and $\forall a \in o^{-1}(\pi(v_i))$

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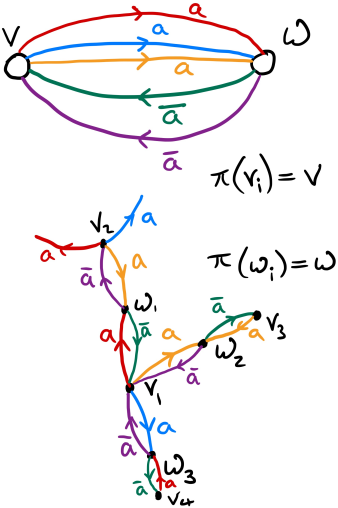

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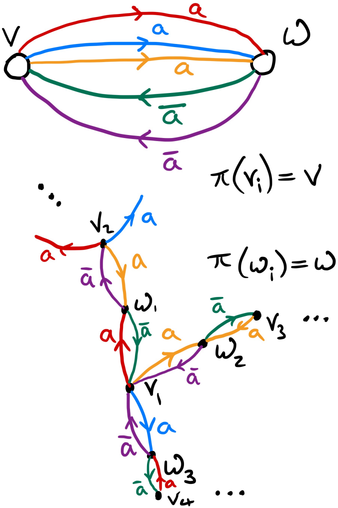

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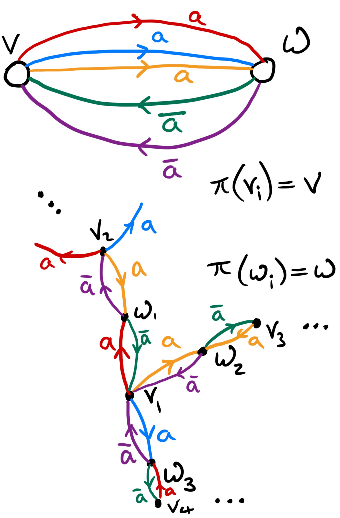
- For Δ-trees $\mathbf{T}, \mathbf{T}'$

  $\exists$ graph isomorphism

  $\theta : T \to T'$ s.t. $\pi' \circ \theta = \pi$



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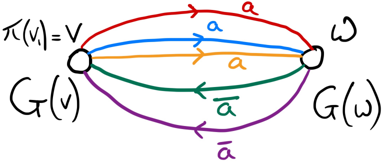

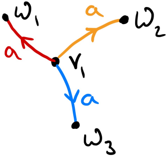

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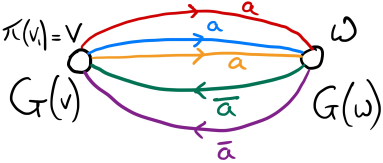

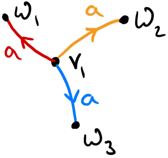

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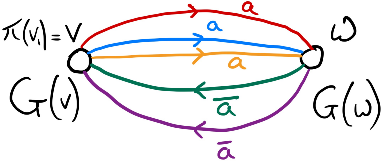

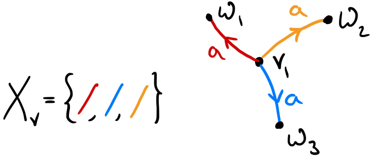

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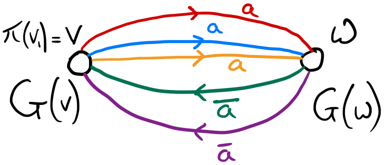

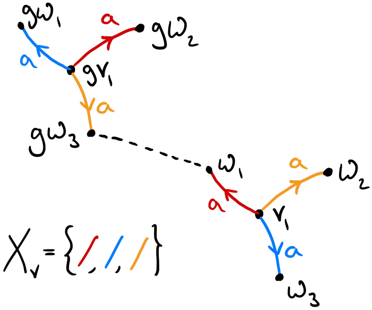

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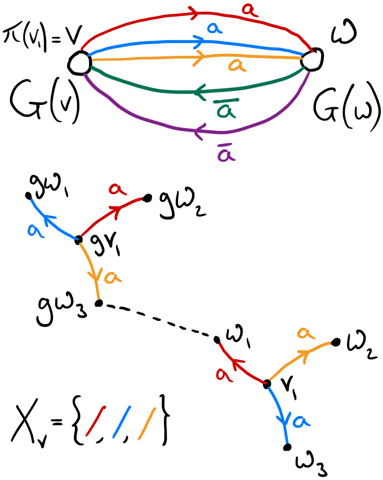

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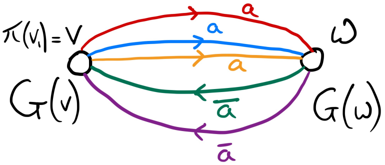

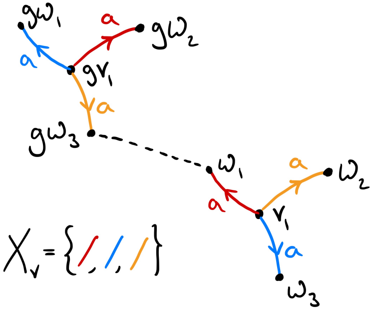

- Choice of $\mathbf{T}$ doesn't matter, results in perm. iso. universal groups.

  Write $\mathbf{U}(\Delta)$ for *the* universal group

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Proof idea: pick representative vertices in T and use their arcs in T as the colours in Δ

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Theorem. (Reid–S.) There is a natural one-to-one correspondence between:

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  Hence: simplicity of G^+

Thank you

Papers to read for more info:

- Marc Burger & Shahar Mozes, ‘Groups acting on trees: from local to global structure’, *Publications mathématiques de l’I.H.É.S.* (2000)
- Colin D. Reid, Simon M. Smith with an appendix by Stephan Tornier, ‘Groups acting on trees with Tits’ independence property (P)’, arXiv:2002.11766
- Simon M. Smith, ‘A product for permutation groups and topological groups’, *Duke Math. J.* (2017)
- Stephan Tornier, ‘Groups Acting on Trees With Prescribed Local Action’, arXiv:2002.09876

Additional slides

Appendix: Proofs for local action diagrams

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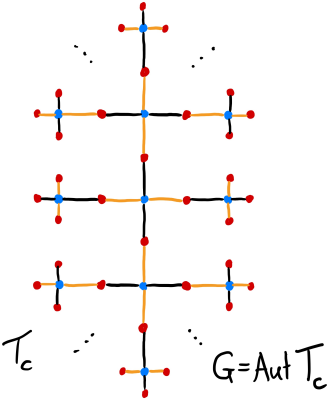

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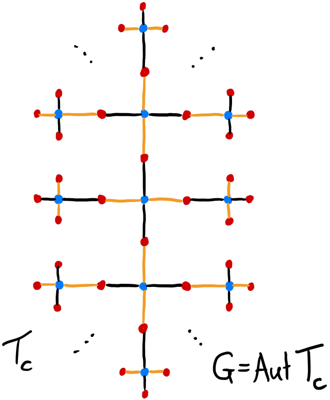

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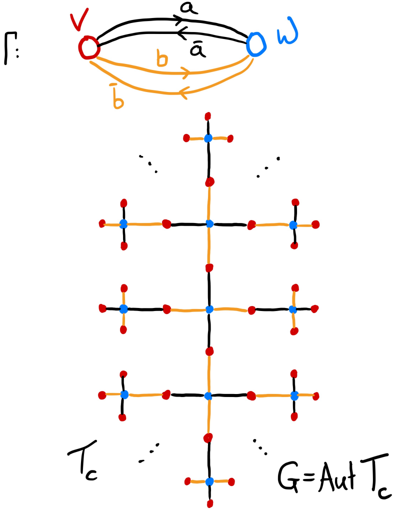

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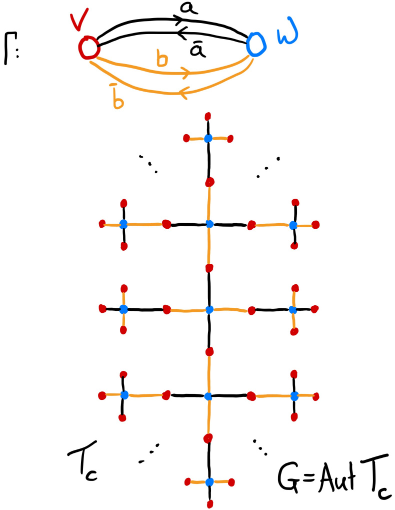

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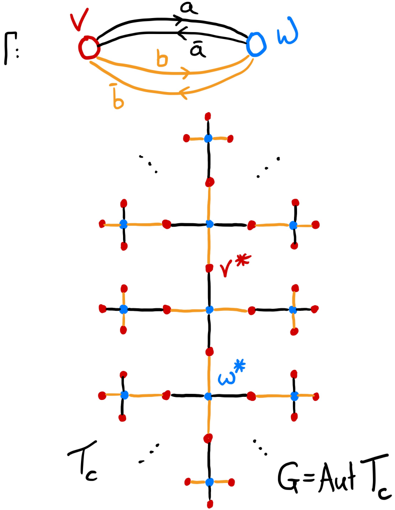

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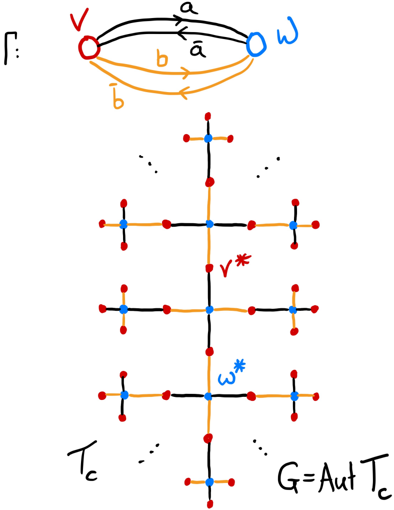

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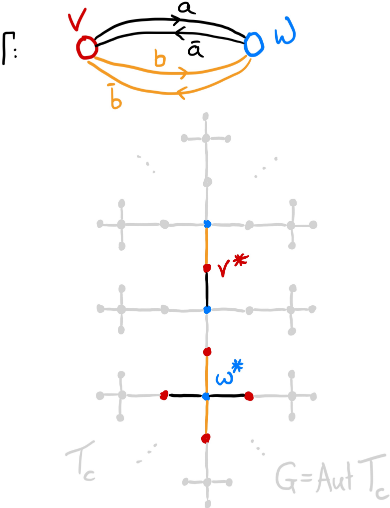

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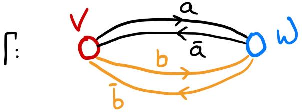

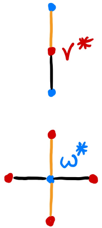

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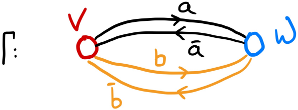

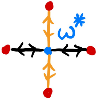

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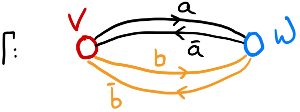

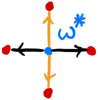

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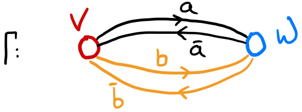

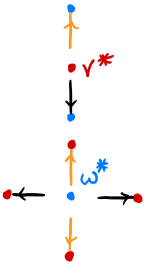

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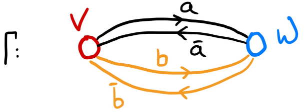

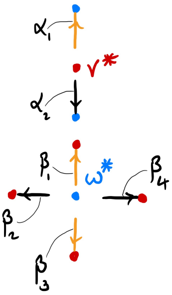

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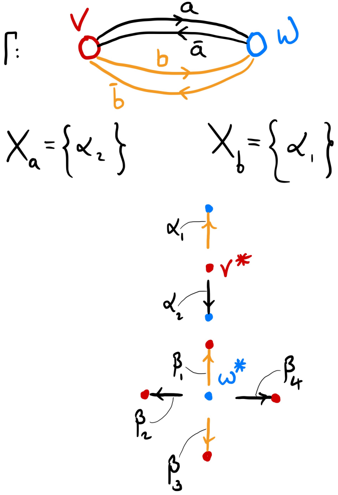

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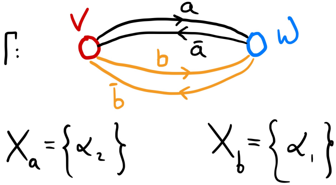

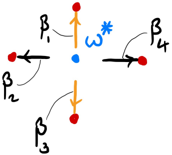

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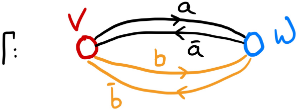

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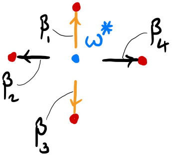

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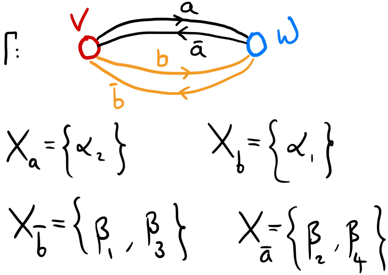

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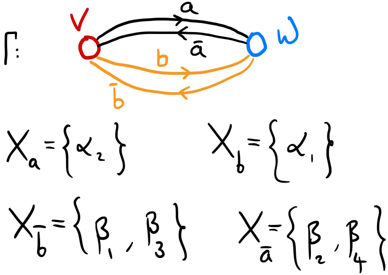

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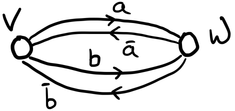

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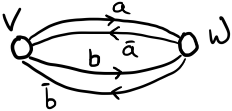

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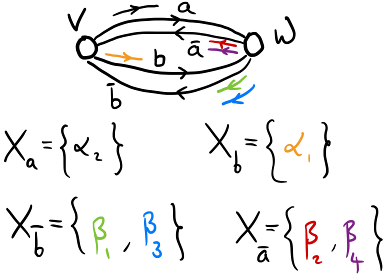

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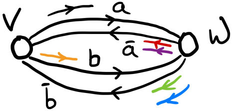

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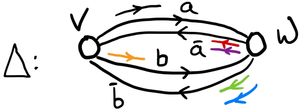

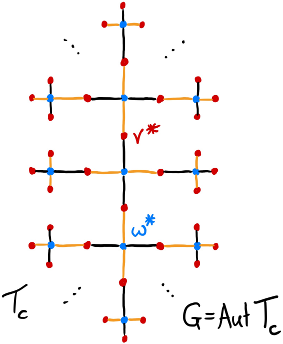

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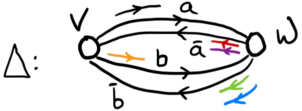

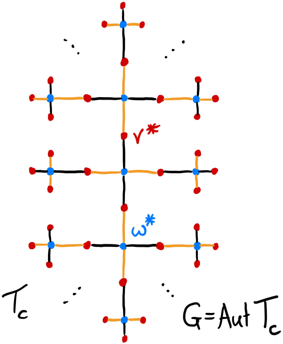

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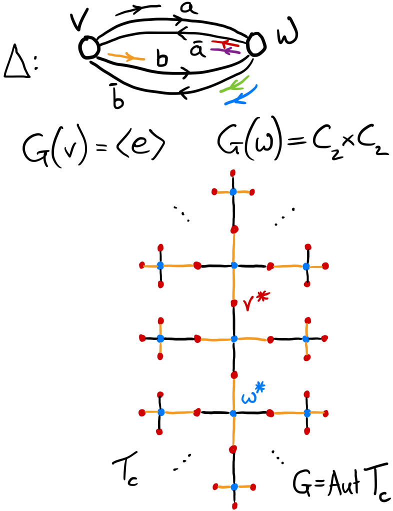

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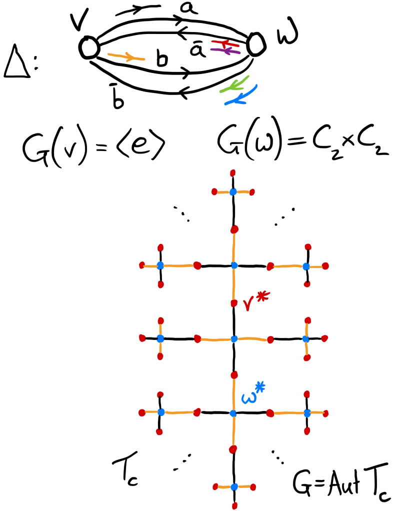
- Finally arc-colour T to form a Δ-tree:
  - $\forall w \in VT$ choose $g_w \in G$ s.t. $g_w w = w^*$
  - $\forall b \in o^{-1}(w)$ set $\mathcal{L}(b) := g_w b$.



Appendix: Further remarks

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- There are only finitely many conjugacy classes in $\mathcal{C}(n, d)$
- The number of conjugacy classes grows rapidly with n and d. For $n = 1$ there are more than just the Burger-Mozes groups.
- Stephan Tornier has an appendix in our paper where he uses GAP to find all (up to conjugacy) (P)-closed groups on T_d the d-regular tree, for $d \leq 5$.