Regularized period of Eisenstein series for unitary groups.

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Notations

- E/F extension of number fields with $[E:F]=2$
- $n \geqslant 1$ an integer
- For any non-degenerate Hermitian space h over E of rank n, we have the following unitary groups:
  - $U'_h = U(h)$ (automorphism group of h);
  - $U_h = U(h) \times U(h \oplus^{\perp} N_{E/F})$ where $N_{E/F}$ is the norm on E.
  - Diagonal embedding $U'_h \hookrightarrow U_h$ given by $g \mapsto (g, \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix})$.

Notations

- $\mathbb{A}$ ring of adèles of F and $|\cdot|_{\mathbb{A}}$ product of normalized local absolute values.
- We fix $P_0 = P_0' \times P_0''$ a minimal parabolic subgroup of U_h.
- We fix $K \subset U_h(\mathbb{A})$ a "good" maximal compact subgroup wrt P_0.
- Let $P = P' \times P''$ be a "standard" parabolic subgroup of U_h i.e. $P_0 \subset P$.
- Spaces of unramified characters: $\mathfrak{a}_P^* = X_F^*(P) \otimes_{\mathbb{Z}} \mathbb{R}$ and its dual $\mathfrak{a}_P$ and its complexified $\mathfrak{a}_{P,\mathbb{C}}^* = \mathfrak{a}_P^* \otimes_{\mathbb{R}} \mathbb{C}$.
- Let $H_P : U_h(\mathbb{A}) \to \mathfrak{a}_P$ be such that $\chi(H_P(pk)) = \log |\chi(p)|_{\mathbb{A}}$ for all $\chi \in X_F^*(P)$, $p \in P(\mathbb{A})$ and $k \in K$.
- We also have: $H_{P'} : U_h'(\mathbb{A}) \to \mathfrak{a}_{P'}$.
- Let $\hat{\tau}_{P'}$ be the characterictic function of the open obtuse chamber ${}^{+}\mathfrak{a}_{P'} \subset \mathfrak{a}_{P'}$ defined by the set of weights $\hat{\Delta}_{P'}$.

Ichino-Yamana truncation operator

- It depends on a "truncation parameter" T namely a point in the positive Weyl chamber in $\mathfrak{a}_{P'_0}$, far away from the walls.
- Denoted by Λ^T_{IY}, it transforms a smooth function φ on $[U_h] = U_h(F)\backslash U_h(\mathbb{A})$ with all its derivatives of uniform moderate growth into a rapidly decreasing one on $[U'_h]$.
- For each $x \in [U'_h]$ we have

$$(\Lambda^T_{IY}\varphi)(x) = \sum_{P'} (-1)^{\dim(\mathfrak{a}_{P'})} \sum_{\delta \in P'(F)\backslash U'_h(F)} \hat{\tau}_{P'}(H_{P'}(\delta x) - T_{P'})\varphi_{P'}(\delta x)$$

with $P'_0 \subset P' \subset U'_h$, $T \in \mathfrak{a}_{P'_0} \mapsto T_{P'} \in \mathfrak{a}_{P'}$ is the natural projection and

$$\varphi_{P'}(x) = \int_{[N_{P''}]} \varphi(nx)\, dn$$

where $N_{P''}$ is the unipotent radical of $P'' = \mathrm{stab}_{U(h \oplus N_{E/F})}(\mathcal{F})$ where $\mathcal{F}$ is the isotropic flag in h s.t. $P' = \mathrm{stab}_{U'_h}(\mathcal{F})$.

Regularized periods of automorphic forms

- Let φ be an automorphic form on $[U_h]$
- We define
$$\mathcal{P}_h^T(\varphi) = \int_{[U'_h]} (\Lambda_{IY}^T \varphi)(x)\, dx$$
- The map $T \mapsto \mathcal{P}_h^T(\varphi)$ coincides with a polynomial exponential:
$$T \mapsto \sum_{\lambda} p_\lambda(T) \exp(\langle \lambda, T \rangle)$$
where λ belongs to a finite subset of $\mathfrak{a}^*_{P'_0,\mathbb{C}}$ and p_λ is a polynomial.
- Under some mild restrictions on the exponents of φ, the polynomial p_0 for $\lambda = 0$ is constant.
- Then, following Ichino-Yamana, we define
$$\mathcal{P}_h(\varphi) = p_0(T).$$

Properties of regularized periods

- $\varphi \mapsto \mathcal{P}_h(\varphi)$ is $U'_h(\mathbb{A})$-invariant.
- We have

$$\mathcal{P}_h(\varphi) = \int_{[U'_h]} \varphi(x)\, dx$$

if the RHS is convergent.

- Let φ be a cuspidal automorphic form on $[U_h]_P = A_M^\infty M(F)N(\mathbb{A})\backslash U_h(\mathbb{A})$ with $P(\mathbb{A}) = A_M^\infty M(\mathbb{A})^1 N(\mathbb{A})$.
- For $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ we have the Eisenstein series

$$E(x, \varphi, \lambda) = \sum_{\delta \in P(F)\backslash U_h(F)} \exp(\lambda(H_P(\delta x)))\varphi(\delta x).$$

- The map

$$\lambda \in \mathfrak{a}_{P,\mathbb{C}}^* \mapsto \mathcal{P}_h(E(\varphi, \lambda))$$

is well-defined outside some hyperplanes.

Some problems

- Gan-Gross-Prasad (GGP) problem. Let $P = MN \subset U_h$ be a parabolic subgroup. Consider a cuspidal subrepresentation σ of M and $\lambda \in i\mathfrak{a}_P^*$. Find a condition under which the linear form

$$\varphi \mapsto \mathcal{P}_h(E(\varphi, \lambda))$$

  does not vanish identically on the automorphic realization of the induced space $\mathrm{Ind}_P^G(\sigma)$.
- Refinement: Ichino-Ikeda problem. Factorize $|\mathcal{P}_h(E(\varphi, \lambda))|^2$ in terms of "natural" local analogues.

We shall give a solution to these two problems for representations σ whose base change to linear groups satisfy some specific conditions. This will give an extension of the original Gan-Gross-Prasad and Ichino-Ikeda conjectures (case $P = U_h$).

Regular Arthur parameter (RAP)

- We set $G_n = \mathrm{Res}_{E/F}\, GL_n(E)$.
- We shall consider Arthur parameters of the following shape: $\pi = \pi_1 \boxtimes \ldots \boxtimes \pi_r$ where
  1. π_k is a cuspidal representation of G_{n_k} where $n_1 + \ldots + n_r = n$.
  2. The representations π_k are two by two distinct.
  3. If $\pi_k = \pi_k^*$ then the Asai L-function $L(s, \pi_{i,k}, \mathrm{As}^{(-1)^{n+1}})$ has a pole at $s = 1$.
  4. If $\pi_k \neq \pi_k^*$, then $\pi_k = \pi_{k'}^*$ for some $k' \neq k$.
- π is discrete if all of its components are self-conjugate dual.
- Let $\pi = \pi_n \boxtimes \pi_{n+1}$ a product of Arthur parameters for $G = G_n \times G_{n+1}$. We shall say that π is regular if no component of π_n can be identified to the contragredient of a component of π_{n+1}.
- We identify a RAP π of G to an automorphic representation $\Pi = \mathrm{Ind}_Q^G(\pi)$ for some parabolic subgroup $Q \subset G$.

Remark Discrete implies regular.

Weak base change

- Let $\Pi = \mathrm{Ind}_Q^G(\pi)$ be a RAP as above. Let
$$\mathfrak{a}_\Pi^* = \{\lambda \in \mathfrak{a}_Q^* = X^*(Q) \otimes \mathbb{R} \mid w\lambda = -\lambda\}$$
where $w \in W^G(M_Q)$ is the permutation that exchanges the components π_k and $\pi_{k'}$ if $\pi_k = \pi_{k'}^*$. We have $\mathfrak{a}_\Pi^* = 0$ iff Π is discrete.
- For $\lambda \in \mathfrak{a}_{\Pi,\mathbb{C}}^*$, we set $\Pi_\lambda = \mathrm{Ind}_Q^G(\pi \otimes \lambda)$.
- Let σ be a cuspidal subrepresentation of the Levi factor M_P of some parabolic subgroup P of U_h.

Definition We shall say that Π is a weak base change of $\Sigma = \mathrm{Ind}_P^{U_h}(\sigma)$ if for almost all places v of F split in E, the local component Π_v is the split base change of Σ_v.

- If it exists, the weak base change is unique (Ramakrishnan).
- If Π is weak base change of Σ, we have an isomorphism $\mathfrak{a}_\Pi^* \simeq \mathfrak{a}_P^*$ and we will not distinguish the two spaces.

Gan-Gross-Prasad conjecture

Theorem 1 Let Π be a regular Arthur parameter and $\lambda \in i\mathfrak{a}_\Pi^*$. The following assertions are equivalent:

1. $L(\frac{1}{2}, \Pi_\lambda) \neq 0$ (Rankin-Selberg L-function);
2. There exists a hermitian space h of rank n, a psg $P \subset U_h$ and a cuspidal representation σ of M_P such that the weak base of $\Sigma = \mathrm{Ind}_P^{U_h}(\sigma)$ is Π and the linear form

$$\varphi \in \Sigma \mapsto \mathcal{P}_h(E(\varphi, \lambda))$$

does not vanish identically.

Remarks In 2, under our assumption on Π, we have for "positive" T

$$\mathcal{P}_h(E(\varphi, \lambda)) = \int_{[U_h']} \Lambda_m^T E(x, \varphi, \lambda) dx$$

and so it is holomorphic on $i\mathfrak{a}_\Pi^* \simeq i\mathfrak{a}_P^*$.

Previous works towards this theorem

They concern the case Π discrete that is that is $P = U_h$ (original GGP conjecture).

- The case $2 \Rightarrow 1$ has been obtained in the work of Ginzburg-Jiang-Rallis, Ichino-Yamana, Jiang-L. Zhang by different methods.
- Here the proof is based on a comparison of relative trace formulae (Jacquet-Rallis strategy).
- Important works in the similar vein first proved the theorem under some local hypotheses on Π that imply that Π is cuspidal (W. Zhang, Xue, Beuzart-Plessis).
- Then Beuzart-Plessis-Liu-Zhang-Zhu proved the theorem in the case Π cuspidal (with no local hypothesis).
- We get the case of non-cuspidal Π (so-called endoscopic) in our joint work with Beuzart-Plessis and Zydor.

A refinement. Notations.

- Let $P \subset U_h$ and σ as before.
- We assume that the weak base change of $\mathrm{Ind}_P^{U_h}(\sigma)$ is a regular Arthur parameter $\Pi = \mathrm{Ind}_Q^G(\pi)$.
- We identify $\mathfrak{a}_P^* \simeq \mathfrak{a}_\Pi^*$.
- For $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$, let $\Sigma_\lambda = \mathrm{Ind}_P^{U_h}(\sigma \otimes \lambda)$ and $\Pi_\lambda = \mathrm{Ind}_Q^G(\pi \otimes \lambda)$.
- We assume that all the local components σ_v are tempered.
- We introduce Tamagawa measures dg on $U_h(\mathbb{A})$ and dh on $U_h'(\mathbb{A})$.
- We fix factorizations $dg = \prod_v dg_v$ and $dh = \prod_v dh_v$ such that for almost all places v

$$\mathrm{vol}(U_h(\mathcal{O}_v), dg_v) = 1 \quad \mathrm{vol}(U_h'(\mathcal{O}_v), dh_v) = 1.$$

  where $\mathcal{O}_v$ is the ring of integers of F_v, the completion of F at v.

Local periods

Let $\Pi_\lambda = \otimes'_v \Pi_{\lambda,v}$, $\Sigma_\lambda = \otimes'_v \Sigma_{\lambda,v}$ and $\eta = \otimes'_v \eta_v$ be the quadratic character attached to E/F. We define the ratio of local L-functions:

$$\mathcal{L}(s, \Sigma_{\lambda,v}) = \prod_{i=1}^{n+1} L(s + i - \frac{1}{2}, \eta_v^i) \frac{L(s, \Pi_{\lambda,v})}{L(s + 1/2, \Sigma_{\lambda,v}, \mathrm{Ad})}$$

We fix an invariant inner product on σ_v which gives an invariant inner product $(\cdot,\cdot)_v$ on Σ_v. We define for a non-zero $\varphi_v \in \Sigma_v$ the local (normalized) period:

$$\mathcal{P}_{h,v}(\varphi_v, \lambda) = \mathcal{L}(\frac{1}{2}, \Sigma_{\lambda,v})^{-1} \int_{U'_h(F_v)} \frac{(\Sigma_{\lambda,v}(h_v)\varphi_v, \varphi_v)_v}{(\varphi_v, \varphi_v)_v} dh_v$$

For $\lambda \in i\mathfrak{a}_P^*$, the integral is convergent and $\mathcal{L}(s, \Sigma_{\lambda,v})$ has neither zero nor pole at $s = \frac{1}{2}$.
For almost all v and non-zero unramified vectors φ_v, we have:

$$\mathcal{P}_{h,v}(\varphi_v, \lambda) = 1.$$

Factorization of periods

Theorem 2 Let $\varphi = \otimes_v \varphi_v \in \mathrm{Ind}_P^{U_h}(\sigma)$ be a non-zero decomposable vector. We have for all $\lambda \in i\mathfrak{a}_P^*$

$$\frac{|\mathcal{P}_h(E(\varphi,\lambda))|^2}{\|\varphi\|_{Pet}^2} = |S_\Pi|^{-1}\mathcal{L}^*(\frac{1}{2},\Sigma_\lambda)\prod_v \mathcal{P}_{h,v}(\varphi_v,\lambda),$$

- $\mathcal{L}(s,\Sigma_\lambda) = \prod_v \mathcal{L}(s,\Sigma_{\lambda,v})$ for $\Re(s) \gg 0$.
- $\mathcal{L}^*(\frac{1}{2},\Sigma_\lambda) = \lim_{s\to\frac{1}{2}}(s-\frac{1}{2})^{-\dim(\mathfrak{a}_P)}\mathcal{L}(s,\Sigma_\lambda)$
- $\|\cdot\|_{Pet}$ is the Petersson norm.
- $S_\Pi = (\mathbb{Z}/2\mathbb{Z})^{\dim(\mathfrak{a}_Q)-2\dim(\mathfrak{a}_P)}$

The case $P = U_h$ has been successively proven by:

- Zhang, Beuzart-Plessis (Π cuspidal + some local hypothesis)
- Beuzart-Plessis-Liu-Zhang-Zhu (Π cuspidal)
- Beuzart-Plessis-C-Zydor for Π non cuspidal.

Applications to Bessel periods

- Let $n \geqslant 0$ and $r \geqslant 1$ be two integers.
- Let h and h_r be two non-degenerate hermitian spaces of resp. rk n and $2r+1$. We assume that

$$h_r = \mathrm{vect}(e_1, \ldots, e_r) \oplus \mathrm{vect}(e_0) \oplus \mathrm{vect}(f_1, \ldots, f_r)$$

  where both $\mathrm{vect}(e_1, \ldots, e_r)$ and $\mathrm{vect}(f_1, \ldots, f_r)$ are maximal isotropic subspaces, with $h_r(e_i, f_j) = \delta_{i,j}$ and $h_r(e_0, e_0) \neq 0$.
- Let $B \subset U(h \oplus^{\perp} h_r)$ be the parabolic subgroup stabilizing the flag

$$\mathrm{vect}(e_1) \subset \mathrm{vect}(e_1, e_2) \subset \ldots \subset \mathrm{vect}(e_1, \ldots, e_r).$$

- Set $U'_h = U(h)$ and $U_h = U(h) \times U(h \oplus^{\perp} h_r)$.
- Diagonal embedding $U'_h \hookrightarrow U_h$.
- Let N_B be the unipotent radical of B. Then U'_h normalizes $\{1\} \times N_B \subset U_h$. We set $S_h = (\{1\} \times N_B) \rtimes U'_h \subset U_h$. This is the Bessel subgroup.

Bessel periods

- We fix an additive character $\psi : E\backslash\mathbb{A}_E \to \mathbb{C}^\times$.
- We get a character of $N_B(\mathbb{A})$ invariant by $U'_h(\mathbb{A})$-conjugation by the formula for $n \in N_B(\mathbb{A})$:

$$\psi(n) = \psi(h_r(ne_2, f_1) + h_r(ne_3, f_2) + \ldots + h_r(ne_r, f_{r-1}) + h_r(ne_0, f_r))$$

- Thus this character extends to a character of $S_h(\mathbb{A})$ trivial on $U'_h(\mathbb{A})$ still denoted by ψ.
- Let σ be a cuspidal subrepresentation of U_h.
- The Bessel period is the linear form

$$\varphi \in \sigma \to \mathcal{B}_h(\varphi) = \int_{[S_h]} \varphi(g)\psi(g)\, dg$$

where $[S_h] = S_h(F)\backslash S_h(\mathbb{A})$.

Remark If $n = 0$ then $U'_h = \{1\}$ and U_h is a quasi-split unitary group of odd rank. Then $S_h \subset U_h$ is a maximal unipotent subgroup and $\mathcal{B}_h(\varphi)$ is a Fourier-Whittaker coefficient of φ.

Gan-Gross-Prasad problem for Bessel periods

Problem. Find a condition on σ under which the Bessel period

$$\varphi \to \mathcal{B}_h(\varphi) = \int_{[S_h]} \varphi(g)\psi(g)\, dg$$

does not vanish identically on σ.

We give an answer for representations whose weak base change to $G = \mathrm{Res}_{E/F}(GL_n(E) \times GL_{n+2r+1}(E))$ is a discrete Arthur parameter $\Pi = \mathrm{Ind}_Q^G(\pi_n \boxtimes \pi_{n+2r+1})$.

Theorem 3 Let Π be a discrete Arthur parameter of $G = G_n \times G_{n+2r+1}$. The following assertions are equivalent:

1. $L(\frac{1}{2}, \Pi) \neq 0$ (Rankin-Selberg L-function);
2. There exists a hermitian form h of rk n, a cuspidal representation σ of U_h such that its weak base to G is Π and the Bessel period

$$\varphi \in \sigma \mapsto \mathcal{B}_h(\varphi)$$

   does not vanish identically.

Remarks

- If $n = 0$ the L-function is trivial and U_h is the quasi-split unitary group of rank $2r + 1$. The theorem is proved by Ginzburg-Rallis-Soudry.
- $2 \Rightarrow 1$ is proved by D.Jiang-L.Zhang.
- In our approach, thm 3 is deduced from thm 1 for some Eisenstein series associated to a maximal parabolic subgroup.
- We expect to get the refinement à la Ichino-Ikeda conjectured by Liu.

Relative trace formula for $U_h = U(h) \times U(h \oplus^{\perp} N_{E/F})$.

This is roughly the following expansion in terms of cuspidal data χ

$$\text{"} \int_{[U_h']^2} \sum_{\gamma \in U_h(F)} f(x^{-1}\gamma y)\, dxdy\text{"} = \sum_{\chi} J_\chi(f)$$

where f is a test function on $U_h(\mathbb{A})$.

Theorem 4 Let χ be the class of a pair (P, σ) such that the weak base change of $\Sigma = \mathrm{Ind}_P^{U_h}(\sigma)$ is a regular Arthur parameter. Then

$$J_\chi^h(f) = \int_{i\mathfrak{a}_P^*} J_{P,\sigma}(\lambda, f) d\lambda$$

where we introduce the relative character

$$J_{P,\sigma}(\lambda, f) = \sum_{\varphi \in ONB \text{ of } \mathrm{Ind}_P^{U_h}(\sigma)} \mathcal{P}(E(\Sigma_\lambda(f)\varphi, \lambda))\overline{\mathcal{P}(E(\varphi, \lambda))}$$