Nonrepetitive coloring

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(joint work with V. Dujmović, G. Joret, B. Walczak, and D. Wood)

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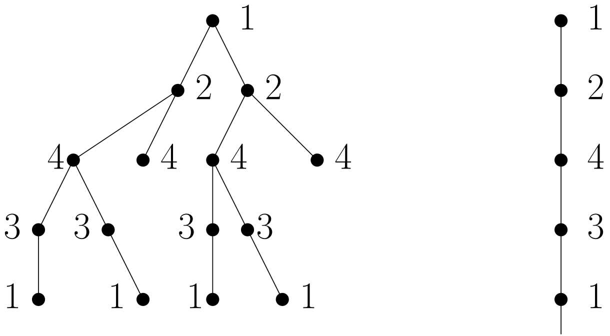

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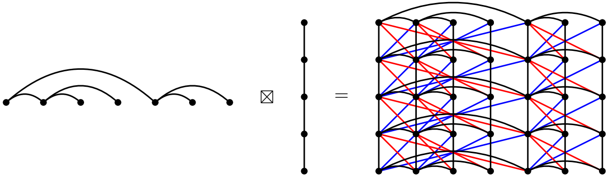
Do planar graphs have bounded nonrepetitive chromatic number?

A product structure in planar graphs

denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A) \times V(B)$ are adjacent if

- $v = w$ and $xy \in E(B)$, or
- $x = y$ and $vw \in E(A)$, or
- $vw \in E(A)$ and $xy \in E(B)$.



Theorem (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood 2019)

Every planar graph is a subgraph of $H \boxtimes P$ for some graph H with treewidth at most 8 and some path P.

Strongly nonrepetitive coloring

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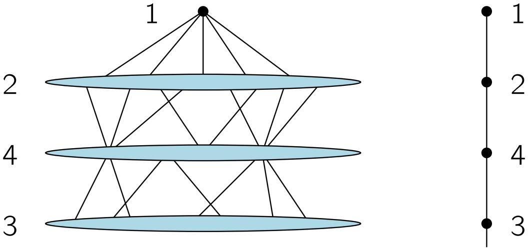

Application to nonrepetitive coloring

Lemma (Dujmović, Esperet, Joret, Walczak, Wood 2019)

For every graph H and every path P, we have $\pi^*(H \boxtimes P) \leq 4\,\pi^*(H)$.

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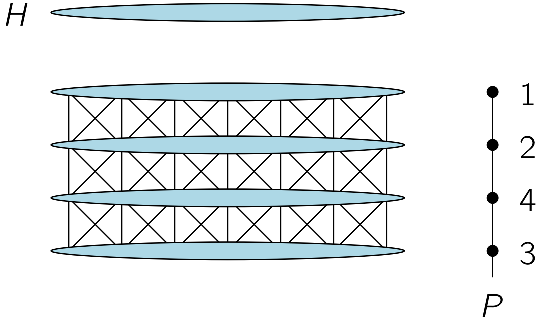

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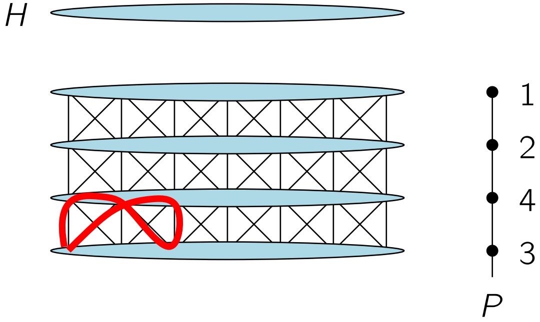

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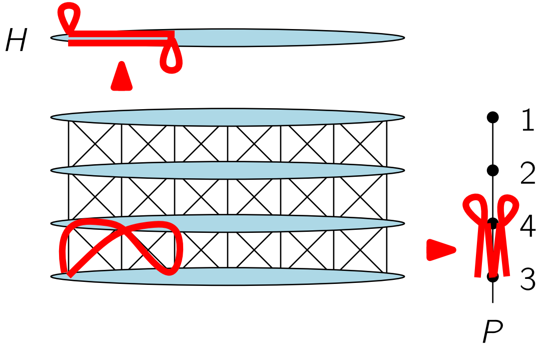

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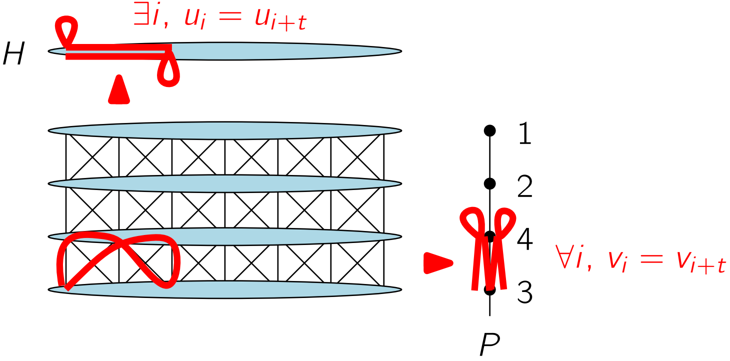

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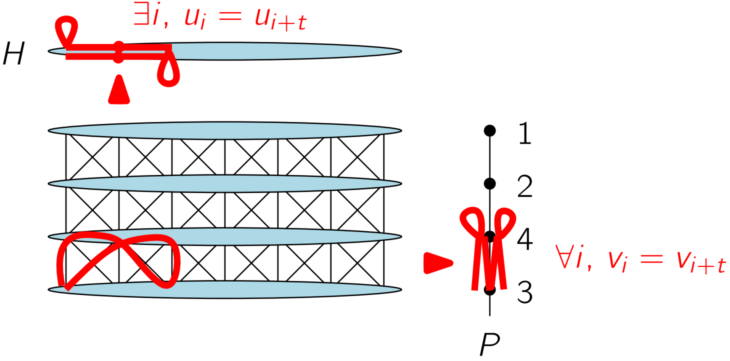

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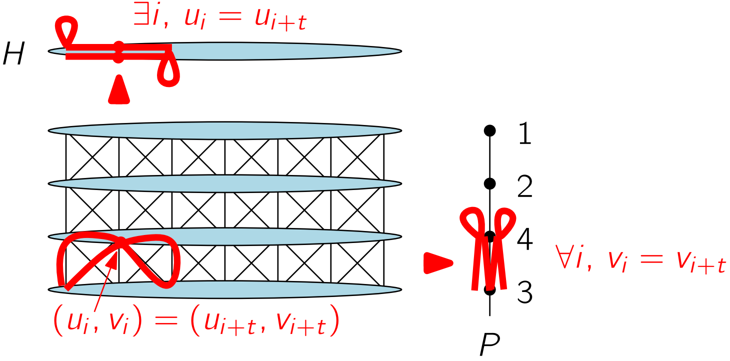

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Graphs of Euler genus g have nonrepetitive chromatic number at most $256 \cdot \max(3, 2g)$.

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Theorem (Dujmović, Esperet, Joret, Walczak, Wood 2019)

Graphs avoiding a fixed minor or topological minor have bounded non-repetitive chromatic number.

Open problems

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- + Several questions of Gwen and David about the connections between bounded expansion and nonrepetitive chromatic number.