An example of a Cartesian differential category from functor calculus

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Tangent Categories and their Applications

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Plan

Describe a project that arose from functor calculus and how it fits into the framework of Cartesian differential categories and tangent categories.

Outline

1. Introduction
2. Abelian functor calculus and directional derivatives
3. Fitting this into a categorical framework
4. A higher order chain rule via tangent categories

Collaborators

Abelian functor calculus and degree 1 chain rule

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Tangent categories and higher order chain rules

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(aka BJORT)

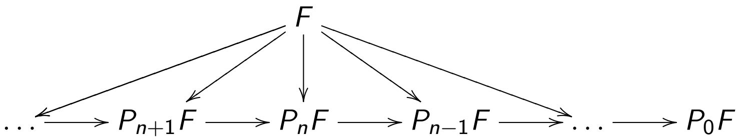
Perspective

- Functor calculus provides a way to approximate functors with "polynomial" functors, like Taylor polynomials.
- Calculus terminology provides an analogy for describing these techniques.
- Cartesian differential and tangent categories validate this analogy and provide new tools for understanding functor calculus.
- This work is rooted in homotopy theory/algebraic topology, where we typically work up to some notion of equivalence (e.g., homotopy equivalence, weak homotopy equivalence, quasi-isomorphism) that's weaker than isomorphism.

Starting Point

Goodwillie, 2003

For a functor of spaces F, there is a tower of functors and natural transformations



such that

- P_nF is an n-excisive functor,
- if F is "nice," the functors P_nF approximate F in a range that increases linearly with n, and
- P_nF is universal (in an appropriate homotopy category) among n-excisive functors with natural transformations from F.

Goodwillie, 2003

F is 1-excisive iff for every strongly (homotopy) cocartesian square

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12}, \end{array}$$

the square

$$\begin{array}{ccc} F(X_0) & \longrightarrow & F(X_1) \\ \downarrow & & \downarrow \\ F(X_2) & \longrightarrow & F(X_{12}) \end{array}$$

is (homotopy) cartesian.

Abelian Functor Calculus – Context

- $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{B} \to \mathcal{A}$ is a functor.
- Eilenberg and Mac Lane (1954) defined "polynomial degree n" functors in this context in terms of *cross effects*.
- Eilenberg and Mac Lane (1951, 1956); and Dold and Puppe (1961) constructed new functors QF (for stable homology of R-modules with coefficients in S) and DF (for derived functors of non-additive functors) that are degree 1 polynomial approximations to F.

Abelian Functor Calculus – Cross effects

An analogy:

For $f : \mathbb{R} \to \mathbb{R}$, f is degree 1 $\Rightarrow f(x) = ax + b$ for some a and b. Then

$$cr_1 f(x) := f(x) - f(0) = ax$$

is linear, and

$$cr_2 f(x, y) = cr_1 f(x + y) - cr_1 f(x) - cr_1 f(y) = 0.$$

For $f : \mathbb{R} \to \mathbb{R}$:

f is degree 2 $\Rightarrow f(x) = ax^2 + bx + c$ for some a, b, and c. Then

$$\begin{aligned} cr_2 f(x,y) &= cr_1 f(x+y) - cr_1 f(x) - cr_1 f(y) \\ &= a(x+y)^2 + b(x+y) - ax^2 - bx - ay^2 - by \\ &= 2axy \end{aligned}$$

is linear in both x and y and

$$\begin{aligned} cr_3 f(x,y,z) &= cr_2 f(x, y+z) - cr_2 f(x,y) - cr_2 f(x,z) \\ &= 2ax(y+z) - 2axy - 2axz = 0. \end{aligned}$$

In fact, f is degree n iff $cr_{n+1} f(x_1, x_2, \ldots, x_{n+1}) = 0$.

Cross Effects

Definition:
For $F : \mathcal{B} \to \mathcal{A}$ where $\mathcal{B}$ and $\mathcal{A}$ are abelian categories, the *nth cross effect functor* $cr_n F : \mathcal{B}^n \to \mathcal{A}$ is defined recursively by

$$F(X) \cong F(0) \oplus cr_1 F(X),$$

$$cr_1 F(X_1 \oplus X_2) \cong cr_1 F(X_1) \oplus cr_1 F(X_2) \oplus cr_2 F(X_1, X_2),$$

and, in general,

$$\begin{aligned} cr_{n-1} F(X_1, \ldots, X_{n-2}, X_{n-1} \oplus X_n) \cong\ & cr_{n-1} F(X_1, \ldots, X_{n-2}, X_{n-1}) \\ & \oplus cr_{n-1} F(X_1, \ldots, X_{n-2}, X_n) \\ & \oplus cr_n F(X_1, \ldots, X_{n-1}, X_n). \end{aligned}$$

Definition:
$F : \mathcal{B} \to \mathcal{A}$ is *degree n* if and only if $cr_{n+1} F \simeq 0$.

Cross Effects

Example

A is an object in an abelian category $\mathcal{A}$, $F : \mathcal{A} \to \mathcal{A}$ with $F(X) = A \oplus X$. Then

$$A \oplus X = F(X) \cong F(0) \oplus cr_1 F(X).$$

Thus,

$$cr_1 F(X) \cong X,$$
$$cr_1 F \cong \mathrm{id}.$$

And,

$$X \oplus Y \cong cr_1 F(X \oplus Y) \cong cr_1 F(X) \oplus cr_1 F(Y) \oplus cr_2 F(X, Y),$$

$$cr_2 F \cong 0.$$

In fact, $cr_n F \cong 0$ for all $n \geq 2$. So F is degree 1.

Abelian Functor Calculus

Theorem (J-McCarthy, 2004)

Given a functor $F : \mathcal{B} \to \mathcal{A}$ between abelian categories $\mathcal{B}$ and $\mathcal{A}$, there exists a Taylor tower of functors and natural transformations

$$\begin{array}{c} F \\ \swarrow \quad \swarrow \quad \downarrow \quad \searrow \quad \searrow \\ \cdots \longrightarrow P_{n+1}F \longrightarrow P_nF \longrightarrow P_{n-1}F \longrightarrow \cdots \longrightarrow P_0F \end{array}$$

such that

- for all $n \geq 0$, P_nF is a degree n functor,
- if F is "nice," the tower converges to F on "nice" objects, and
- P_nF is universal (in an appropriate homotopy category) among degree n functors with natural transformations from F.

Linearization and Directional Derivatives

The functor $F : \mathcal{B} \to \mathcal{A}$ is *linear* iff F is degree one and reduced ($F(0) = 0$).

The *linearization* of $F : \mathcal{B} \to \mathcal{A}$ is

$$D_1 F := P_1(cr_1 F) \simeq \mathrm{hofiber}(P_1 F \to P_0 F).$$

Example

For $F : X \mapsto X \oplus A$, $D_1 F : X \mapsto X$.

Linearization and Directional Derivatives

Reminder:

For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, a point $x \in \mathbb{R}^n$ and a direction $v \in \mathbb{R}^n$, the *directional derivative of f at x in the direction v* is

$$\nabla f(v; x) = \lim_{t \to 0} \frac{1}{t} \left[f(x + tv) - f(x) \right].$$

Definition:

For $F : \mathcal{B} \to \mathcal{A}$, and $X, V \in \mathcal{B}$, the *directional derivative of F* is

$$\nabla F(V; X) := D_1 F(X \oplus -)(V) \simeq D_1^V (\ker(F(X \oplus V) \to F(X))).$$

$\ker(F(X \oplus V) \to F(X)) \Leftrightarrow f(x + tv) - f(x)$

$D_1^V \Leftrightarrow$ linearize w.r.t. $V \Leftrightarrow \lim_{t \to 0} \frac{1}{t}$

Theorem - Chain Rule for Directional Derivatives (J-McCarthy, 2004; BJORT, 2018)

For a composable pair of functors F and G,

$$\nabla(F \circ G) \simeq \nabla F(\nabla G; G).$$

Cartesian Differential Categories

Definition (Blute-Cockett-Seely, 2009):

A Cartesian left-additive category with a differential operator ∇

$$\frac{F : \mathcal{B} \to \mathcal{A}}{\nabla F : \mathcal{B} \times \mathcal{B} \to \mathcal{A}}$$

is a *Cartesian differential category* if ∇ satisfies properties 1-7 for morphisms F, G, and objects X, Y, Z, W, V:

Linear 1 $\nabla(F \oplus G)(V; X) = \nabla F(V; X) \oplus \nabla G(V; X)$.

Linear 2 $\nabla F(V \oplus W; X) = \nabla F(V; X) \oplus \nabla F(W; X)$ and $\nabla F(0; X) = 0$.

Identity $\nabla \mathrm{id}(V; X) = V$.

Products $\nabla \langle F, G \rangle (V; X) = \langle \nabla F(V; X), \nabla G(V; X) \rangle$.

Chain Rule $\nabla(F \circ G)(V; X) = \nabla F(\nabla G(V; X); G(X))$.

Partials $\nabla(\nabla F)((Z; 0); (0; X)) = \nabla F(Z; X)$.

Mixed ∂ $\nabla(\nabla F)((Z; W); (V; X)) = \nabla(\nabla F)((Z; V); (W; X))$.

Cartesian Differential Categories

Goal:

Prove that the directional derivative ∇ endows a category whose objects are abelian categories and morphisms are functors between abelian categories with the structure of a Cartesian differential category.

Question:

What's the correct category?

Context-revisited

1. We're really working with functors $F : \mathcal{B} \to Ch\mathcal{A}$, where $Ch\mathcal{A}$ is the category of chain complexes concentrated in degrees ≥ 0.

$$A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} \cdots \leftarrow \mathcal{A}_{n-1} \xleftarrow{\partial_n} \ldots$$

$$\partial_{n-1} \circ \partial_n = 0$$

2. We're working up to chain homotopy equivalence: $F \simeq G$ means for a each object X, there is a chain homotopy equivalence between $F(X)$ and $G(X)$.

Cartesian Differential Categories – Issues

- For $F : \mathcal{B} \to Ch\mathcal{A}$ and $G : \mathcal{C} \to Ch\mathcal{B}$, how do we form $F \circ G$?
- Applying F degreewise does not preserve chain complexes or chain homotopy equivalences.
- The chain rule only holds up to chain homotopy equivalence.

Constructing the right category

A monad on *AbCat*

- *AbCat* is the category whose objects are abelian categories and morphisms are functors between abelian categories.
- *Ch* is a monad on *AbCat*:

$$\mathcal{A} \mapsto Ch\mathcal{A}$$

For $F : \mathcal{B} \to \mathcal{A}$, $ChF : Ch\mathcal{B} \to Ch\mathcal{A}$ is the composite

$$Ch\mathcal{B} \xrightarrow{\simeq} \mathcal{B}^{\Delta^{\mathrm{op}}} \xrightarrow{F_*} \mathcal{A}^{\Delta^{\mathrm{op}}} \xrightarrow{\simeq} Ch\mathcal{A}.$$

where the left and right arrows are the Dold-Kan equivalence.
- The unit is $\deg_0 : \mathcal{A} \to Ch\mathcal{A}$, $A \mapsto (A \leftarrow 0 \leftarrow 0 \leftarrow \dots)$.
- The multiplication is the total complex functor: $\mathrm{Tot} : ChCh\mathcal{A} \to Ch\mathcal{A}$.

Constructing the right category

$AbCat_{Ch}$ is the Kleisli category associated to $AbCat$ and Ch

- Objects are abelian categories.
- morphisms are functors $\mathcal{B} \to Ch\mathcal{A}$
- identity morphism is $\deg_0 : \mathcal{A} \to Ch\mathcal{A}$
- the composition of $G : \mathcal{C} \to Ch\mathcal{B}$ and $F : \mathcal{B} \to Ch\mathcal{A}$ is the composite

$$\mathcal{C} \xrightarrow{G} Ch\mathcal{B} \xrightarrow{ChF} ChCh\mathcal{A} \xrightarrow{Tot} Ch\mathcal{A}.$$

Cartesian Differential Categories

Theorem (BJORT).

∇ and $AbCat_{Ch}$ satisfy:

1. $\nabla(F \oplus G)(V; X) \cong \nabla F(V; X) \oplus \nabla G(V; X)$.
2. $\nabla F(V \oplus W; X) \simeq \nabla F(V; X) \oplus \nabla F(W; X)$ and $\nabla F(0; X) \cong 0$.
3. $\nabla \mathrm{id}(V; X) \cong V$.
4. $\nabla \langle F, G \rangle (V; X) \cong \langle \nabla F(V; X), \nabla G(V; X) \rangle$.
5. $\nabla(F \circ G)(V; X) \simeq \nabla F(\nabla G(V; X); G(X))$.
6. $\nabla(\nabla F)((Z; 0); (0; X)) \cong \nabla F(Z; X)$.
7. $\nabla(\nabla F)((Z; W); (V; X)) \simeq \nabla(\nabla F)((Z; V); (W; X))$.

$\simeq$ denotes chain homotopy equivalences, $\cong$ denotes isomorphims.

Corollary (BJORT)

The homotopy category $HoAbCat_{Ch}$ (objects same as $AbCat_{Ch}$, morphisms $[\mathcal{B}, \mathcal{A}]$ are chain homotopy classes of functors) with the directional derivative ∇ is a Cartesian differential category.

Higher Order Chain Rules:

For composable functions of real vector spaces (or Banach spaces), Huang, Marcantognini and Young (2006) defined higher order directional derivatives, Δ_n, and derived a chain rule:

$$\Delta_n(f \circ g) = \Delta_n f(\Delta_n g, \Delta_{n-1} g, \ldots, \Delta_1 g; g).$$

When g is a function of a single variable, this yields a chain rule for ordinary derivatives:

$$(f \circ g)^{(n)}(x) = \Delta_n f(g^{(n)}(x), g^{(n-1)}(x), \ldots, g'(x); g).$$

In degree 1, this is

$$\nabla(f \circ g) = \nabla f(\nabla g; g).$$

Question:

Is there an analogous notion of a higher order directional derivative and a higher order chain rule for abelian functor calculus?

Higher Order Directional Derivatives

For $F : \mathcal{B} \to \mathcal{A}$ and objects $V_1, V_2, \ldots, V_n, X$ in $\mathcal{B}$, the *higher order directional derivatives* of F are given by

$$\begin{aligned} \Delta_0 F(X) &:= F(X), \\ \Delta_1 F(V_1; X) &:= \nabla F(V_1; X) \\ \Delta_2 F(V_2, V_1; X) &:= \nabla(\Delta_1 F)((V_2, V_1); (V_1, X)), \end{aligned}$$

and, in general,

$$\begin{aligned} &\Delta_n F(V_n, \ldots, V_1; X) \\ &\qquad := \nabla(\Delta_{n-1} F)((V_n, \ldots, V_2; V_1); (V_{n-1}, \ldots, V_1; X)). \end{aligned}$$

Higher Order Chain Rule

Theorem (BJORT)

For a composable pair of functors F and G,

$$\Delta_n(F \circ G)(V_n, \dots, V_1; X)$$

is chain homotopy equivalent to

$$\Delta_n F(\Delta_n G(V_n, \dots, V_1; X), \dots, \Delta_2 G(V_2, V_1; X), \Delta_1 G(V_1; X), G(X)).$$

Higher Order Chain Rule – proof

- R. Cockett and G. Cruttwell (2014): Cartesian differential categories are *tangent categories*.
- Tangent categories are equipped with an endofunctor T encoding essential properties of tangent bundles.
- $T : HoAbCat_{Ch} \to HoAbCat_{Ch}$ is defined for
  - an object $\mathcal{A}$ by $T(\mathcal{A}) = \mathcal{A} \times \mathcal{A}$,
  - a morphism $F : \mathcal{B} \to Ch\mathcal{A}$, $TF : \mathcal{B} \times \mathcal{B} \to Ch\mathcal{A} \times Ch\mathcal{A}$ via

$$TF(V, X) = \langle \nabla F(V; X), F(X) \rangle.$$

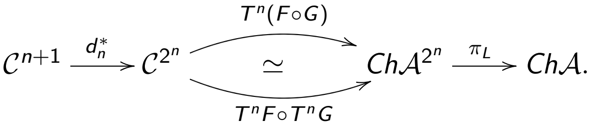
Higher Order Chain Rule – proof

Goal:

$$\Delta_n(F \circ G)(V_n, \ldots, V_1; X) \simeq$$

$$\Delta_n F(\Delta_n G(V_n, \ldots, V_1; X), \ldots, \Delta_1 G(V_1; X); G(X))$$

Key: Use the diagram of functors



When applied to $(V_n, \ldots, V_1; X)$, composition along the top yields

$$\Delta_n(F \circ G)(V_n, \ldots, V_1; X),$$

and composition along the bottom yields

$$\Delta_n F(\Delta_n G(V_n, \ldots, V_1; X), \ldots, \Delta_1 G(V_1; X); G(X)).$$

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