Dual Tangent Structures for Infinity-Toposes

BIRS/FMCS Workshop on Tangent Categories and Applications

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I. ∞-Toposes [Toën-Vezzosi, Rezk, Lurie]

An <u>∞-topos</u> is an ∞-category $\mathcal{X}$ s.t. there is an adjunction of the form

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift right, leftarrow, "f"', phantom] & \mathcal{P}(C) \arrow[l, shift right, "f"'] \end{tikzcd}$$

$$\mathcal{X} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{P}(C) \quad (\perp)$$

∞-cat. of ∞-groupoids

where (i) $\mathcal{P}(C) = \mathrm{Fun}(C^{op}, \mathcal{S})$ is the ∞-category of presheaves (of spaces) on a small ∞-category C

(ii) the left adjoint f preserves finite limits

(iii) the right adjoint g is fully faithful and accessible (preserves κ-filtered colimits for some κ)

$\mathcal{X}$ is a left exact accessible localization of $\mathcal{P}(C)$

Examples of ∞-Toposes

① $\mathcal{P}(\mathcal{C})$ for a small ∞-category $\mathcal{C}$

e.g. $* = \mathcal{P}(\emptyset)$, $\mathcal{S} = \mathcal{P}(1)$

② for top. space X : $\mathrm{Shv}(X) \subseteq \mathcal{P}(\mathcal{O}(X))$:

$F : \mathcal{O}(X)^{op} \to \mathcal{S}$ s.t. for open covers $\mathcal{U} = \bigcup_i \mathcal{U}_i$:

$$F(\mathcal{U}) \simeq \operatorname{holim}\left(\prod_i F(\mathcal{U}_i) \rightrightarrows \prod_{i,j} F(\mathcal{U}_i \cap \mathcal{U}_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

e.g. $\mathrm{Shv}(\emptyset) \simeq *$, $\mathrm{Shv}(\bullet) \simeq \mathcal{S}$

③ for any 1-topos $\mathcal{T}$: $\mathcal{T}_\infty \subseteq \mathcal{P}(\mathcal{T}^{op})$

(4) for any ∞-topos $\mathfrak{X}$, small ∞-category C:

$$\mathrm{Fun}(C, \mathfrak{X}) \text{ is an } \infty\text{-topos}$$

(5) if C has pushouts and terminal object

excisive functors $\rightsquigarrow$ $\mathrm{Exc}(C, \mathfrak{X}) \subseteq \mathrm{Fun}(C, \mathfrak{X})$

is an ∞-topos: inclusion has left adjoint P_1

left exact (preserves finite lim)

(6) for ∞-topos $\mathfrak{X}$: the tangent bundle

$$T(\mathfrak{X}) = \mathrm{Exc}(\mathcal{S}_{\mathrm{fin},*}, \mathfrak{X})$$

is an ∞-topos.

e.g. $T(\mathcal{S}) = \{\text{parametrized spectra}\}$

Geometric vs. Algebraic Perspectives [Anel–Joyal]

An ∞-topos is:

1. an ∞-category with similar properties to $\mathcal{S}$ the ∞-category of ∞-groupoids (Giraud axioms)

   ALGEBRAIC

2. a generalized topological space

   GEOMETRIC

Morphisms of ∞-Toposes

A <u>geometric morphism</u> $\mathcal{X} \to \mathcal{Y}$ between ∞-toposes is an adjunction

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift right, "g"'] \arrow[r, phantom, "\perp"] & \mathcal{Y} \arrow[l, shift right, "f"'] \end{tikzcd}$$

s.t. f preserves finite limits.

There is a subcategory $\mathrm{Topos}_\infty \subseteq \mathrm{Cat}_\infty$ of ∞-toposes and geometric morphisms.

We can also identify $\mathrm{Topos}_\infty^{op} = \mathrm{Logos}_\infty$ with a subcategory of Cat_∞, whose morphisms are the left exact left adjoints.

Examples of geometric morphisms

① $f : X \to Y$ continuous :

$$\begin{tikzcd} \mathrm{Shv}(X) \arrow[r, shift right, "f_*"'] \arrow[r, phantom, "\perp"] & \mathrm{Shv}(Y) \arrow[l, shift right, "f^*"'] \end{tikzcd}$$

② For any ∞-topos $\mathcal{X}$, there are unique geometric morphisms

$$\begin{tikzcd} * \arrow[r, phantom, "\perp"] \arrow[r, shift right] & \mathcal{X} \arrow[l, bend right] \arrow[r, phantom, "\perp"] \arrow[r, shift right] & \mathcal{S} \arrow[l, bend right] \end{tikzcd}$$

$\mathrm{Shv}(\emptyset)$ initial in Topos_∞

$\mathrm{Shv}(\bullet)$ terminal in Topos_∞

∞-Categories of Points

preserve colim + finite lim

$\mathcal{X}$: ∞-topos

$$\mathrm{Pt}(\mathcal{X}) := \underline{\mathrm{Hom}}_{\mathrm{Topos}_\infty}(\mathcal{S}, \mathcal{X}) := \mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$$

e.g. $\mathrm{Pt}(\mathrm{Shv}(X)) \simeq$ poset of "points" of X under specialization

$$\mathrm{Pt}(\mathcal{P}(C)) \simeq \mathrm{Ind}(C^{op})$$

$\mathrm{Pt} : \mathrm{Topos}_\infty \longrightarrow \mathrm{Cat}_\infty^\omega$ ← accessible ∞-categories with small filtered colimits and filtered-colim-preserving functors

has left adjoint $Q : \mathrm{Cat}_\infty^\omega \longrightarrow \mathrm{Topos}_\infty$:

$$\mathcal{C} \longmapsto \mathrm{Fun}^\omega(\mathcal{C}, \mathcal{S})$$ ← functors preserve filt. colim

II Goodwillie Tangent Structure

Theorem

There is a tangent structure on the ∞-category

$$\mathrm{Logos}_\infty = \mathrm{Topos}_\infty^{op}$$

Lurie's tangent bundle on $\mathcal{X}$

with tangent bundle

$$T(\mathcal{X}) := \mathrm{Exc}(\mathcal{S}_{fin,*}, \mathcal{X}),$$

the restriction to $\mathrm{Topos}_\infty^{op} \subseteq \mathrm{Cat}_\infty^{diff}$ of the Goodwillie tangent structure on $\mathrm{Cat}_\infty^{diff}$.

[Bauer-Burke-C.]

Remark

tensor product of presentable ∞-categories (Lurie)

$$T(\mathcal{X}) \simeq T(\mathcal{S}) \boxtimes \mathcal{X}$$

where $\boxtimes$ is the coproduct in $\mathrm{Topos}_\infty^{op}$ / product in Topos_∞.

Compare: there is a tangent structure on CRing given by

$$T(R) = R[x]/(x^2) = T(\mathbb{Z}) \otimes R$$

coproduct CRing

The Goodwillie tangent structure is corepresented by an infinitesimal structure on $T(\mathcal{S})$: a monoidal functor

$$(\mathrm{Weil}, \otimes, \mathbb{N}) \longrightarrow (\mathrm{Topos}_\infty^{op}, \boxtimes, \mathcal{S}) ; \quad W \mapsto T(\mathcal{S})$$

that preserves the tangent limits.

III Geometric Tangent Structure

Theorem

There is a tangent structure $\mathcal{U}$ on Topos_∞ given by

$$\mathcal{U}(\mathfrak{X}) = \mathfrak{X}^{T(J)}$$

(exponential object in Topos_∞)

represented by the infinitesimal object $T(J)$.

i.e.

$$\begin{tikzcd} \mathrm{Topos}_\infty \arrow[r, shift right=1.5ex, "\mathcal{U} = (\ )^{T(J)}"'] & \mathrm{Topos}_\infty \arrow[l, shift right=1.5ex, "T^{op} = - \boxtimes T(J)"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

Compare: There is a tangent structure on $\mathrm{CRing}^{op} = \mathrm{AffSch}$

given by $\mathcal{U}(R) = R^{T(\mathbb{Z})} = \mathrm{Sym}\,\Omega_R$

Proof [Cockett–Cruttwell, 5.17]: sufficient to show exponentials exist for each $\mathcal{T}_n(\mathcal{S})$

[Anel–Lejay, Lurie SAG 21.1]

$\mathcal{X}$ is exponentiable iff $\mathrm{Ind}(\mathcal{X}) \xrightarrow{\mathrm{colim}} \mathcal{X}$ has a left adjoint

iff $\mathcal{X}$ is a retract of a compactly-generated ∞-category

by functors that preserve filtered colimits

($\mathcal{X}$ is presentable compactly-assembled/continuous).

$\mathcal{T}_n(\mathcal{S})$ is compactly-generated, so exponentiable.

Affine and Injective ∞-Toposes

affine ∞-topos : $\mathcal{P}(C^{\mathrm{lex}})$ where C^{lex} freely adds

free ∞-logos   finite limits to small ∞-category C

injective ∞-topos : retract of affine ∞-topos

projective ∞-logos

Theorem (Anel-Lejay, Lurie)

There are equivalences of ∞-categories:

or continuous

$$\mathrm{Pt} : \mathrm{InjTopos}_\infty \underset{\longleftarrow}{\overset{\sim}{\longrightarrow}} \mathrm{Cat}_\infty^{\mathrm{pres,comp\text{-}ass}} : Q$$

injective ∞-topoi

Theorem (C.)

Pt, Q are equivalences of tangent ∞-categories

$$\Big(\mathrm{InjTopos}_\infty,\ \underbrace{\mathcal{U}}_{\text{geometric}}\Big) \underset{Q}{\xleftarrow{\ \simeq\ }} \Big(\mathrm{Cat}_\infty^{\mathrm{pres,comp\text{-}ass}},\ \underbrace{T}_{\text{Goodwillie}}\Big)$$

Proof Lemma (Anel–Lejay, Cune) $\mathcal{C} \in \mathrm{Cat}_\infty^{\mathrm{pres,comp\text{-}ass}}$

$$\underline{\mathrm{Hom}}_{\mathrm{Topos}_\infty}(\mathcal{X}, Q\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{X}$$

Proof: If $\mathcal{C}$ is comp. gen: $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C}_0)$

Then $Q\mathcal{C} = \mathrm{Fun}^{\omega}(\mathcal{C}, \mathcal{S}) = \mathrm{Fun}(\mathcal{C}_0, \mathcal{S}) = \mathcal{P}(\mathcal{C}_0^{op})$.

$$\underline{\mathrm{Hom}}_{\mathrm{Topos}_\infty}(\mathcal{X}, \mathcal{P}(\mathcal{C}_0^{op})) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}_0^{op}, \mathcal{X}) = \mathrm{Fun}^{\lim}(\mathrm{Ind}(\mathcal{C}_0)^{op}, \mathcal{X}) = \mathcal{C} \boxtimes \mathcal{X}.$$

$$\underline{\mathrm{Hom}}_{\mathrm{Topos}_\infty}(\mathcal{X}, UQ\mathcal{C}) \simeq \underline{\mathrm{Hom}}_{\mathrm{Topos}_\infty}(T\mathcal{X}, Q\mathcal{C})$$

$$\simeq \mathcal{C} \otimes T\mathcal{X}$$

$$\simeq \mathcal{C} \otimes T\mathcal{S} \otimes \mathcal{X}$$

$$\simeq T\mathcal{C} \otimes \mathcal{X}$$

$$\simeq \underline{\mathrm{Hom}}_{\mathrm{Topos}_\infty}(\mathcal{X}, Q(T\mathcal{C}))$$

So $\quad UQ \simeq QT$

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