The World of Differential Categories: A Tutorial on Cartesian Differential Categories

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Tutorial Talk for BIRS2021

Thanks Kristine, Geoff, and Robin for organizing the conference and the invitation.

The Differential Category World: The Four Tomes

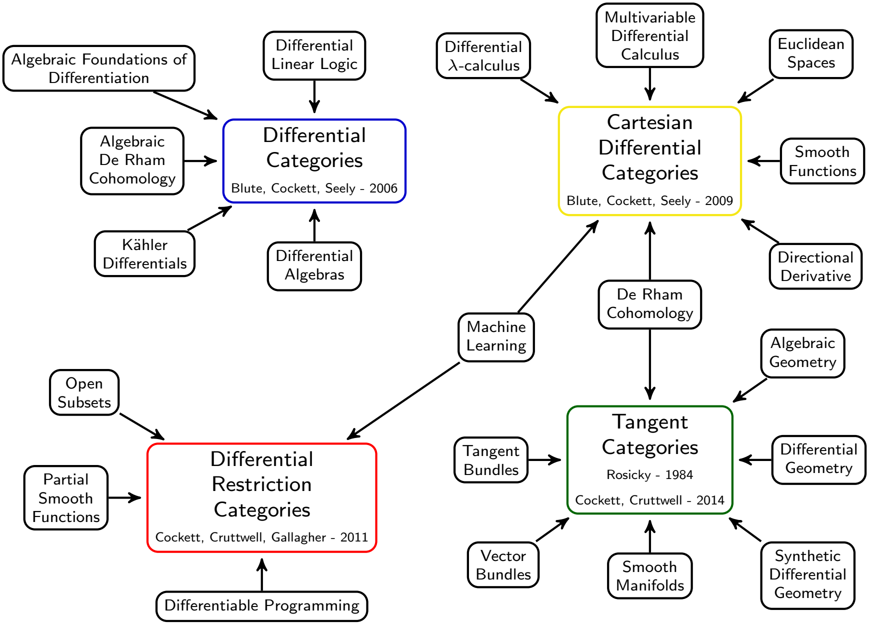
Differential Categories
Blute, Cockett, Seely - 2006

Cartesian Differential Categories
Blute, Cockett, Seely - 2009

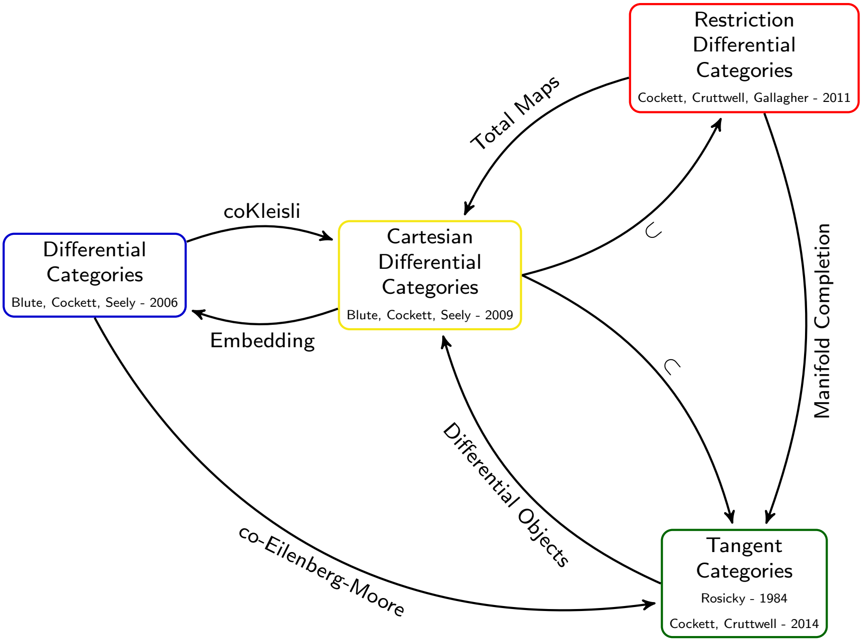
Differential Restriction Categories
Cockett, Cruttwell, Gallagher - 2011

Tangent Categories
Rosicky - 1984
Cockett, Cruttwell - 2014

The Differential Category World: A Taster



The Differential Category World: It's all connected!



Today's Story: Cartesian Differential Categories

Cartesian Differential Categories:

- Formalize differentiation in multivariable calculus of Euclidean spaces.
- Provide the categorical semantics of the differential λ-calculus.

  T. Ehrhard, L. Regnier **The differential λ-calculus.** (2003)

Main Reference:

R. Blute, R. Cockett, R.A.G. Seely, **Cartesian Differential Categories**

Cartesian Differential Categories - Definition

A **Cartesian differential category** is:

1. A Cartesian left additive category;
2. With a differential combinator.

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2. With a differential combinator.

Cartesian Left Additive Category - Definition

A **left additive category** is a category $\mathbb{X}$ which is *skew-enriched* over commutative monoids:

Campbell, A., 2018. **Skew-enriched categories**.

Explicitly, every homset is a commutative monoid, so we can add maps and have zero maps:

$$+ : \mathbb{X}(A, B) \times \mathbb{X}(A, B) \to \mathbb{X}(A, B) \qquad 0 \in \mathbb{X}(A, B)$$

such that composition preserves the addition in the following sense:

$$(f + g) \circ x = f \circ x + g \circ x \qquad 0 \circ x = 0$$

A map f is **additive** if $f \circ (x + y) = f \circ x + f \circ y$ and $f \circ 0 = 0$.

A **Cartesian left additive category** (CLAC) is a left additive category with finite products such that the projection maps $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ are additive.

Cartesian Left Additive Categories - Examples

Example

- Every category with finite biproducts is a CLAC where every map is additive. For example, VEC_k the category of k-vector spaces and k-linear maps is a CLAC.

- VEC_k^ω the category of k-vector spaces and arbitrary set functions is a CLAC, where the sum of set functions is defined point-wise $(f+g)(x) = f(x) + g(x)$.

- Let Poly_k be the Lawvere theory of polynomials, that is, the category whose objects are $n \in \mathbb{N}$ and where a map $P : n \to m$ is a tuple of polynomials:

$$P = \langle p_1(\vec{x}), \ldots, p_m(\vec{x}) \rangle \qquad p_i(\vec{x}) \in R[x_1, \ldots, x_n]$$

  Then Poly_k is a CLAC (where $n \times m = n + m$).

- Let SMOOTH be the category of smooth real functions, that is, the category whose objects are the Euclidean vector spaces $\mathbb{R}^n$ and whose maps are smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, which is actually an m-tuple of smooth functions:

$$F = \langle f_1, \ldots, f_m \rangle \qquad f_i : \mathbb{R}^n \to \mathbb{R}$$

  Then SMOOTH is a CLAC. Note that $\mathsf{Poly}_{\mathbb{R}}$ is a sub-CLAC of SMOOTH.

A **Cartesian differential category** is:

1. A Cartesian left additive category;
2. With a **differential combinator**.

Differential Combinator - Definition

A **differential combinator** on a Cartesian left additive category $\mathbb{X}$ is a combinator D, which is a family of functions $\mathbb{X}(A, B) \to \mathbb{X}(A \times A, B)$, which written as an inference rule:

$$\frac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$$

Before giving the axioms, let's look at some examples!

Differential Combinator - Main Example

Example

SMOOTH is a Cartesian differential category where the differential combinator is defined as the directional derivative of a smooth function. A smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$ is in fact a tuple:

$$F = \langle f_1, \ldots, f_m \rangle$$

of smooth functions $f_i : \mathbb{R}^n \to \mathbb{R}$. Then the Jacobian matrix of F at vector $\vec{x} \in \mathbb{R}^n$ is the matrix $\mathbf{J}(F)(\vec{x})$ of size $m \times n$ whose coordinates are the partial derivatives of the f_i:

$$\mathbf{J}(F)(\vec{x}) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \frac{\partial f_1}{\partial x_2}(\vec{x}) & \ldots & \frac{\partial f_1}{\partial x_n}(\vec{x}) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}) & \frac{\partial f_2}{\partial x_2}(\vec{x}) & \ldots & \frac{\partial f_2}{\partial x_n}(\vec{x}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{x}) & \frac{\partial f_m}{\partial x_2}(\vec{x}) & \ldots & \frac{\partial f_m}{\partial x_n}(\vec{x}) \end{bmatrix}$$

So for a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$, its derivative $\mathsf{D}[F] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ is then defined as:

$$\mathsf{D}[F](\vec{x}, \vec{y}) := \mathbf{J}(F)(\vec{x}) \cdot \vec{y} = \left\langle \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(\vec{x}) y_i, \ldots, \sum_{i=1}^{n} \frac{\partial f_m}{\partial x_i}(\vec{x}) y_i \right\rangle$$

where $\cdot$ is matrix multiplication and $\vec{y}$ is seen as a $n \times 1$ matrix. For example, Let $f(x_1, x_2) = x_1^3 x_2$.

$$\mathsf{D}[f]\left((x_1, x_2), (y_1, y_2)\right) = 3x_1^2 x_2 y_1 + x_1^3 y_2$$

Cartesian Differential Categories - Other Main Examples

Example

Any category with finite biproduct $\oplus$ is a CDC, where for a map $f : A \to B$:

$$\mathsf{D}[f] := \; A \oplus A \xrightarrow{\quad \pi_1 \quad} A \xrightarrow{\quad f \quad} B$$

For example, VEC_k is a CDC where $\mathsf{D}[f](x, y) = f(y)$.

Example

POLY_k is a CDC where for a map $P : n \to m$ with $P = \langle p_1(\vec{x}), \ldots, p_m(\vec{x}) \rangle$, $\mathsf{D}[P] : n \times n \to m$ is:

$$\mathsf{D}[P] := \left\langle \sum_{i=1}^{n} \frac{\partial p_1(\vec{x})}{\partial x_i} y_i, \ldots, \sum_{i=1}^{n} \frac{\partial p_m(\vec{x})}{\partial x_i} y_i \right\rangle$$

where $\sum\limits_{i=1}^{n} \frac{\partial p_i(\vec{x})}{\partial x_i} y_i \in R[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Note that $\mathsf{POLY}_{\mathbb{R}}$ is a sub-CDC of SMOOTH.

Differential Combinator - Definition

A **differential combinator** on a Cartesian left additive category $\mathbb{X}$ is a combinator D, which is a family of functions $\mathbb{X}(A, B) \to \mathbb{X}(A \times A, B)$, which written as an inference rule:

$$\frac{f : A \to B}{\mathsf{D}[f] : A \times A \to B}$$

To help us with the axioms, we will use the following notation/proto-term logic:

$$\mathsf{D}[f](a, b) := \frac{\mathsf{d}f(x)}{\mathsf{d}x}(a) \cdot b$$

Example

The notation comes from SMOOTH: $\mathsf{D}[F](\vec{x}, \vec{y}) := \mathbf{J}(F)(\vec{x}) \cdot \vec{y}$.

Remark

There is a sound and complete term logic for Cartesian differential categories. In short: anything we can prove using the term logic, holds in any Cartesian differential category. So doing proofs in the term logic is super useful!

CD.1 - Additivity of Combinator & CD.2 - Additivity in Second Argument

- Additivity of Combinator:

$$\mathrm{D}[f+g] = \mathrm{D}[f] + \mathrm{D}[g] \qquad \mathrm{D}[0] = 0$$

$$\frac{\mathrm{d}f(x)+g(x)}{\mathrm{d}x}(a)\cdot b = \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot b + \frac{\mathrm{d}g(x)}{\mathrm{d}x}(a)\cdot b \qquad \frac{\mathrm{d}0}{\mathrm{d}x}(a)\cdot b = 0$$

- Additivity in Second Argument

$$\mathrm{D}[f]\circ\langle a, b+c\rangle = \mathrm{D}[f]\circ\langle a, b\rangle + \mathrm{D}[f]\circ\langle a, c\rangle \qquad \mathrm{D}[f]\circ\langle x, 0\rangle = 0$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot(b+c) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot b + \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot c \qquad \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a)\cdot 0 = 0$$

CD.3 - Identities + Projections & CD.4 - Pairings

- Identities + Projections

$$\mathrm{D}[1] = \pi_1 \qquad \mathrm{D}[\pi_i] = \pi_i \circ \pi_1$$

$$\frac{\mathrm{d}x}{\mathrm{d}x}(a) \cdot b = b \qquad \frac{\mathrm{d}x_i}{\mathrm{d}(x_0, x_1)}(a_0, a_1) \cdot (b_0, b_1) = b_i$$

- Pairings

$$\mathrm{D}[\langle f, g\rangle] = \langle \mathrm{D}[f], \mathrm{D}[g]\rangle$$

$$\frac{\mathrm{d}\langle f(x), g(x)\rangle}{\mathrm{d}x}(a) \cdot b = \left\langle \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b, \frac{\mathrm{d}g(x)}{\mathrm{d}x}(a) \cdot b \right\rangle$$

Example

In SMOOTH, if $F = \langle f_1, \dots, f_n\rangle$, then $\mathrm{D}[F](\vec{x}, \vec{y}) := \langle \mathrm{D}[f_1](\vec{x}, \vec{y}), \dots, \mathrm{D}[f_n](\vec{x}, \vec{y})\rangle$.

CD.5 - Chain Rule

Chain Rule:

$$\mathrm{D}[g \circ f] = \mathrm{D}[g] \circ \langle f \circ \pi_0, \mathrm{D}[f] \rangle$$

$$\frac{\mathrm{d}g(f(x))}{\mathrm{d}x}(a) \cdot b = \frac{\mathrm{d}g(x)}{\mathrm{d}x}(f(a)) \cdot \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b \right)$$

CD.6 - Linearity in Second Argument & CD.7 - Symmetry

$$\cfrac{\cfrac{f : A \to B}{\mathrm{D}[f] : A \times A \to B}}{\mathrm{D}\left[\mathrm{D}[f]\right] : (A \times A) \times (A \times A) \to B}$$

- Linearity in Second Argument

$$\mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle a, 0, 0, b\rangle = \mathrm{D}[f] \circ \langle a, b\rangle$$

$$\frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(y) \cdot z}{\mathrm{d}(y, z)}(a, 0) \cdot (0, b) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b$$

- Symmetry

$$\mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle\langle a, b\rangle, \langle c, d\rangle\rangle = \mathrm{D}\left[\mathrm{D}[f]\right] \circ \langle\langle a, c\rangle, \langle b, d\rangle\rangle$$

$$\frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(y) \cdot z}{\mathrm{d}(y, z)}(a, b) \cdot (c, d) = \frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(y) \cdot z}{\mathrm{d}(y, z)}(a, c) \cdot (b, d)$$

More on these axioms soon!

Cartesian Differential Categories - Definition

A **Cartesian differential category** is:

1. A Cartesian left additive category;
2. With a differential combinator.

$$\frac{f : A \to B}{\mathrm{D}[f] : A \times A \to B}$$

Before we give some more examples: let's see what we can do within a CDC!

Partial Derivatives I

Suppose we have a map $f : A \times B \to C$ and we only want to differentiate with respect to A.

We can zero out in $\mathsf{D}[f] : (A \times B) \times (A \times B) \to C$ to obtain a partial derivative!

Define the partial derivative $\mathsf{D}_0[f] : (A \times B) \times A \to C$ as follows:

$$\mathsf{D}_0[f] := (A \times B) \times A \xrightarrow{(1_A \times 1_B) \times \langle 1_A, 0 \rangle} (A \times B) \times (A \times B) \xrightarrow{\mathsf{D}[f]} C$$

$$\mathsf{D}_0[f](a, b, c) := \frac{\mathrm{d}f(x, b)}{\mathrm{d}x}(a) \cdot c := \frac{\mathrm{d}f(x, y)}{\mathrm{d}(x, y)}(a, b) \cdot (c, 0)$$

Similarly, define the partial derivative $\mathsf{D}_1[f] : (A \times B) \times B \to C$ as follows:

$$\mathsf{D}_1[f] := (A \times B) \times B \xrightarrow{(1_A \times 1_B) \times \langle 0, 1_B \rangle} (A \times B) \times (A \times B) \xrightarrow{\mathsf{D}[f]} C$$

$$\mathsf{D}_1[f](a, b, d) := \frac{\mathrm{d}f(a, y)}{\mathrm{d}y}(b) \cdot d := \frac{\mathrm{d}f(x, y)}{\mathrm{d}(x, y)}(a, b) \cdot (0, d)$$

You can also do this with maps $f : A_0 \times \ldots \times A_n \to B$.

Partial Derivatives II

A consequence of symmetry rule, CD.7, is that for $f : A \times B \to C$, doing the partial derivative with respect to A then B is the same as doing the partial derivative with respect to B then A.

$$\frac{\mathrm{d}\,\frac{\mathrm{d}f(x,y)}{\mathrm{d}y}(b)\cdot d}{\mathrm{d}x}(a)\cdot c = \frac{\mathrm{d}\,\frac{\mathrm{d}f(x,y)}{\mathrm{d}x}(a)\cdot c}{\mathrm{d}y}(b)\cdot d$$

Additivity in the second argument, CD.2, tells us that for $f : A \times B \to C$, $\mathrm{D}[f]$ is the sum of the partial derivatives!

$$\begin{aligned}
\frac{\mathrm{d}f(x,y)}{\mathrm{d}(x,y)}(a,b)\cdot(c,d) &= \frac{\mathrm{d}f(x,y)}{\mathrm{d}(x,y)}(a,b)\cdot((c,0)+(0,d)) \\
&= \frac{\mathrm{d}f(x,y)}{\mathrm{d}(x,y)}(a,b)\cdot(c,0) + \frac{\mathrm{d}f(x,y)}{\mathrm{d}(x,y)}(a,b)\cdot(0,d) \\
&= \frac{\mathrm{d}f(x,b)}{\mathrm{d}x}(a)\cdot c + \frac{\mathrm{d}f(a,y)}{\mathrm{d}y}(b)\cdot d
\end{aligned}$$

Example

For a smooth map $f : \mathbb{R}^n \to \mathbb{R}$, $\mathrm{D}[f]$ is the sum of its partial derivatives:

$$\mathrm{D}[f] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \qquad \mathrm{D}[f](\vec{v},\vec{w}) := \mathbf{J}(f)(\vec{v})\cdot\vec{w} = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(\vec{v})w_i$$

Linear Maps I

In a Cartesian differential category, there is a natural notion of **linear maps**. A map $f : A \to B$ is said to be linear if:

$$\mathsf{D}[f] := A \times A \xrightarrow{\pi_1} A \xrightarrow{f} B$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot b = f(b)$$

Example

- In a category with finite biproducts, every map is linear (by definition!).
- In POLY_k, $P = \langle p_1, \ldots, p_m \rangle$ is linear if each $p_i \in k[x_1, \ldots, x_n]$ is a polynomial of degree 1, that is, a sum of the form $p_i = \sum_{j=1}^{n} a_j x_j$.
- In $\mathsf{SMOOTH}_{\mathbb{R}}$, a smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$ is linear in the Cartesian differential sense precisely when it is $\mathbb{R}$-linear in the classical sense:

$$F(s\vec{x} + t\vec{y}) = sF(\vec{x}) + tF(\vec{y})$$

for all $s, t \in \mathbb{R}$ and $\vec{x}, \vec{y} \in \mathbb{R}^n$.

- Linear $\Rightarrow$ Additive, but not necessarily the converse!
  (But in the above examples: Additive $\Rightarrow$ Linear)
- Identity maps and projection maps are linear by CD.3

Linear Maps II

A map $f : A \times B \to C$ can also be linear in its second argument if it is linear with respect to its partial derivative:

$$\mathsf{D}_1[f] := \ (A \times B) \times B \xrightarrow{\pi_0 \times 1} A \times B \xrightarrow{f} C$$

$$\frac{\mathrm{d}f(a,y)}{\mathrm{d}y}(b) \cdot c = f(a,c)$$

The linearity in the second argument rule, CD.6, says that for any $f : A \to B$, $\mathsf{D}[f]$ is linear in its second argument:

$$\frac{\mathrm{d}\frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot y}{\mathrm{d}y}(b) \cdot c = \frac{\mathrm{d}f(x)}{\mathrm{d}x}(a) \cdot c$$

Example

For a smooth map $f : \mathbb{R}^n \to \mathbb{R}$, $\mathsf{D}[f]$ is linear in its second argument:

$$\mathsf{D}[f] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \qquad \mathsf{D}[f](\vec{v}, \vec{w}) := \mathbf{J}(f)(\vec{v}) \cdot \vec{w} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{v}) w_i$$

Cartesian Differential Categories - Other Examples

Example

Every model of the differential λ-calculus induces a Cartesian differential category. Conversly, every Cartesian differential category which is Cartesian closed such that the evaluation maps are linear in their second argument gives rises to a model of the differential λ-calculus.

Manzonetto, G., 2012. **What is a Categorical Model of the Differential and the Resource λ-Calculi?**.

Example

Bauer, Johnson, Osborne, Riehl, and Tebbe (BJORT) constructed an Abelian functor calculus model of a Cartesian differential category.

Bauer, K., Johnson, B., Osborne, C., Riehl, E. and Tebbe, A., 2018. **Directional derivatives and higher order chain rules for abelian functor calculus.**

Example

There is a couniversal construction of Cartesian differential categories, known as the Faa di Bruno construction, that is, for every Cartesian left additive category $\mathbb{X}$ there is a cofree Cartesian differential category over $\mathbb{X}$.

Cockett, J.R.B. and Seely, R.A.G., 2011. **The Faa di bruno construction**.

Cartesian Differential Categories - Other Applications

Example

Study and solve differential equations, and also study exponential functions, trigonometric functions, hyperbolic functions, etc.

- Cockett, R., Cruttwel, G., Lemay, J-S. P., **Differential equations in a tangent category I: Complete vector fields, flows, and exponentials**.
- Cockett, R., Lemay, J-S.P., **Exponential Functions for Cartesian Differential Categories**.

Example

There is a notion of integration for Cartesian differential categories.

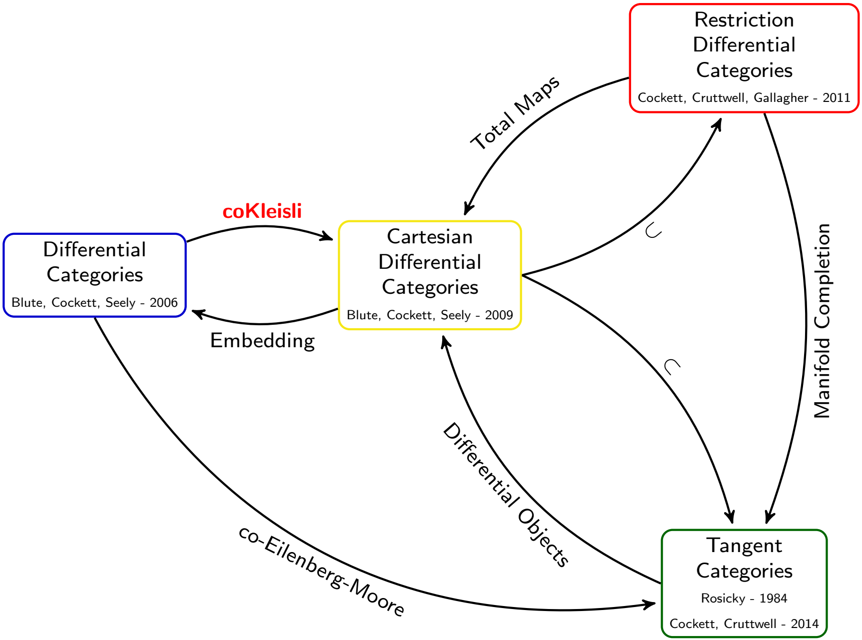
- Lemay, J-S.P., **Cartesian Integral Categories and Contextual Integral Categories**.

Example

Machine learning algorithms and differentiable programming languages via **reverse differentiation**.

- Cockett, R., Cruttwell, G., Gallagher, J., Lemay, J. S. P., MacAdam, B., Plotkin, G., & Pronk, D. (2020). *Reverse derivative categories.*
- Wilson, P., & Zanasi, F. Reverse *Derivative Ascent: A Categorical Approach to Learning Boolean Circuits.*
- Cruttwell, G., Gallagher, J., & Pronk, D. *Categorical semantics of a simple differential programming language.*
- Cruttwell, G., Gavranovic, B., Ghani, N., Wilson, P., & Zanasi, F. *Categorical Foundations of Gradient-Based Learning.*

The Differential Category World: It's all connected!



Differential Categories - Smooth Maps

Every differential category has a notion of a *smooth map*.

A smooth map $A \to B$ is a coKleisli map, that is, a map $!A \to B$.

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Example

Let's consider the example where $!(\mathbb{R}^n) := \mathrm{Sym}(\mathbb{R}^n) \cong \mathbb{R}[x_1, \ldots, x_n]$.

$$p : \mathbb{R}^n \to \mathbb{R} \qquad p \text{ is a polynomial function}$$

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$$\dfrac{\dfrac{p : \mathbb{R}^n \to \mathbb{R} \qquad p \text{ is a polynomial function}}{p \in \mathrm{Sym}(\mathbb{R}^n)}}{\hat{p} : \mathbb{R} \to \mathrm{Sym}(\mathbb{R}^n) \qquad \text{linear map in } \mathrm{VEC}_{\mathbb{R}} \text{ where } \hat{p}(1) = p}$$

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The coKleisli Category of a Differential Category I

Consider a differential category $\mathbb{X}$ with a coalgebra modality !:

$$!A \xrightarrow{\delta} !!A \qquad !A \xrightarrow{\varepsilon} A$$

$$!A \xrightarrow{\Delta} !A \otimes !A \qquad !A \xrightarrow{\mathsf{e}} I$$

and deriving transformation:

$$!A \otimes A \xrightarrow{\mathsf{d}} !A$$

and finite products $\times$ (which are actually biproducts by the additive structure of $\mathbb{X}$).

Let $\mathbb{X}_!$ be the coKleisli category and we are going to use interpretation brackets $[\![-]\!]$.

$$\frac{f : A \to B \text{ in } \mathbb{X}_!}{[\![f]\!] : !A \to B}$$

$$[\![1]\!] = \ !A \xrightarrow{\varepsilon} A$$

$$[\![g \circ f]\!] = \ !A \xrightarrow{\delta} !!A \xrightarrow{!([\![f]\!])} !B \xrightarrow{[\![g]\!]} C$$

So how do we make $\mathbb{X}_!$ into a Cartesian differential category?

The coKleisli Category of a Differential Category II

For the product structure:

- On objects, $A \times B$
- Projections:

$$[\![\pi_i]\!] := \ !(A_0 \times A_1) \xrightarrow{\quad\varepsilon\quad} A_0 \times A_1 \xrightarrow{\quad\pi_i\quad} A_i$$

For a comonad on a category with finite products, the coKleisli category has finite products.

For the additive structure:

- The sum of maps: $[\![f + g]\!] := [\![f]\!] + [\![g]\!]$
- Zero maps: $[\![0]\!] := 0$

For a comonad on an additive category, the coKleisli category is ONLY a left additive category, because coKleisli composition does not preserve the additive structure. However, every coKleisli map of the form $f \circ \varepsilon$ is additive.

For a comonad on an additive category with finite products, the coKleisli category is a Cartesian left additive category.

The coKleisli Category of a Differential Category III

Recall that last time we defined the differential of $[\![f]\!] : !A \to B$ as:

$$!A \otimes A \xrightarrow{\quad \mathsf{d} \quad} !A \xrightarrow{\quad [\![f]\!] \quad} B$$

But this is not a coKleisli map!

The differential combinator $[\![\mathsf{D}[f]]\!] : !(A \times A) \to B$ is defined as follows:

$$!(A \times A) \xrightarrow{\Delta} !(A \times A) \otimes !(A \times A) \xrightarrow{!(\pi_0)\otimes !(\pi_1)} !A \otimes !A \xrightarrow{1 \otimes \varepsilon} !A \otimes A \xrightarrow{\mathsf{d}} !A \xrightarrow{[\![f]\!]} B$$

Theorem

For a differential category with finite products, its coKleisli category is a Cartesian differential category.

Every coKleisli map of the form $f \circ \varepsilon$ is linear! (This is an if and only if when $!0 \cong I$)

Example

Consider the differential category VEC_k^{op} with $!(V) = \mathsf{Sym}(V)$ from last time. Then POLY_k is a sub-CDC of the coKleisli category $\left(\mathsf{VEC}_k^{op}\right)_{\mathsf{Sym}}$. More explicit examples are described in:

Bucciarelli, A. and Ehrhard, T. and Manzonetto, G. **Categorical models for simply typed resource calculi**.
which include the relational model and the finiteness space model

The other direction: Cartesian differential storage categories

Blute, R., Cockett, J.R.B. and Seely, R.A., 2015. **Cartesian differential storage categories**.

"... it was not obvious how to pass from Cartesian differential categories back to monoidal differential categories.This paper provides natural conditions under which the linear maps of a Cartesian differential category form a monoidal differential category. ... The purpose of this paper is to make precise the connection between the two types of differential categories. "

Main idea: While not every Cartesian differential category is the coKleisli category of a differential category, **Cartesian differential storage categories** are precisely the coKleisli categories of differential categories.

Theorem

A differential category with finite products and the Seely isomorphisms ($!(A \times B) \cong !A \otimes !B$ and $!0 \cong I$), it's coKleisli category is a Cartesian differential storage category. Conversely, for a Cartesian differential storage category, its category of linear maps form a differential category with finite products and the Seely isomorphisms.

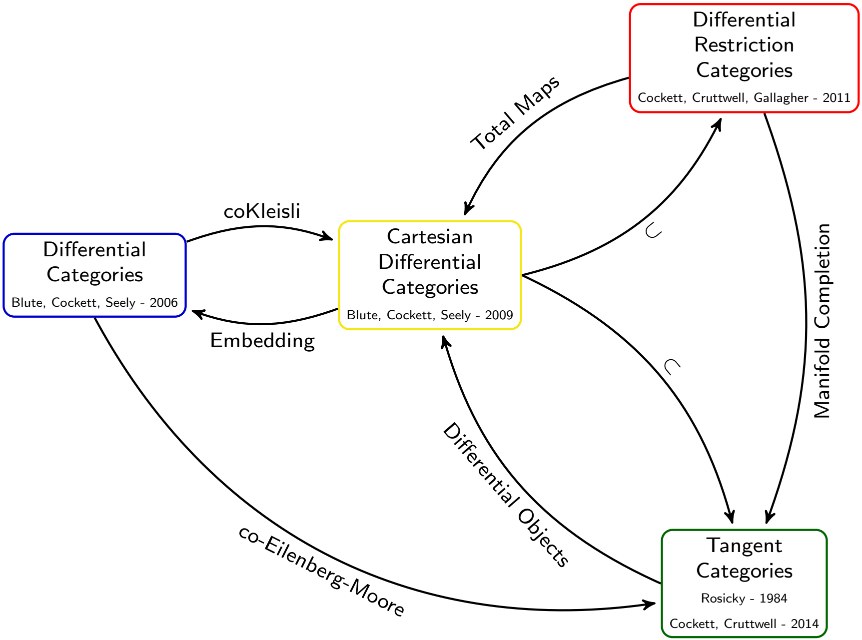
The other direction: Embedding

Garner, R, and Lemay, J-S P. **Cartesian differential categories as skew enriched categories**.

Theorem

Every Cartesian differential category embeds into the coKleisli category of a differential category.

The Differential Category World: It's all connected!



A quick word on Differential Restriction Categories

A **restriction category** is a category equipped with a restriction operator

$$\frac{f : A \to B}{\overline{f} : A \to A}$$

where you should think of $\overline{f}$ as capturing the domain of definition of f. Restriction categories allow us to work with partially defined functions.

Lack, S., and Cockett, R. **Restriction Categories (I - III)**.

A **differential restriction category** is NAIVELY a Cartesian differential category with a restriction operator such that the differential operator and restriction operator are compatible.

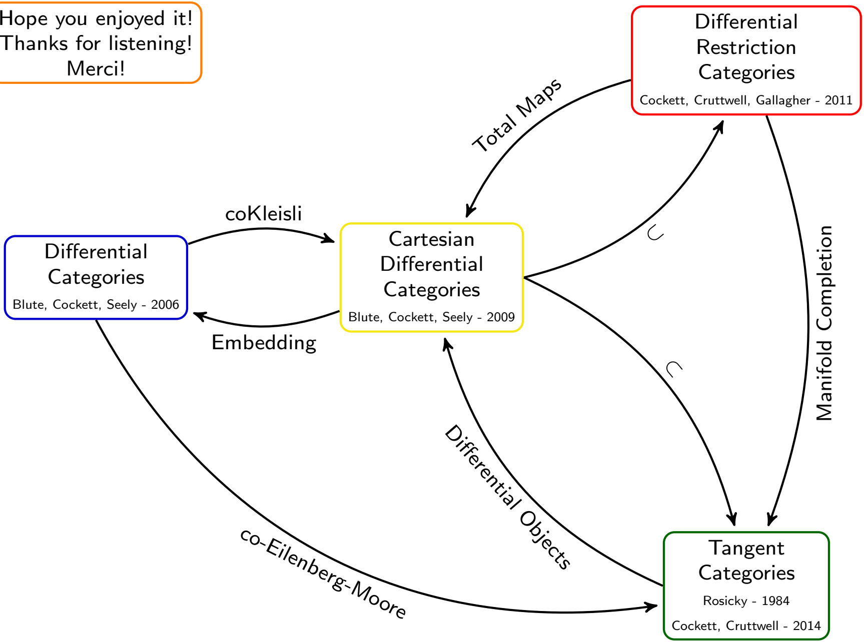
Cockett, R., Cruttwell, G., and Gallagher, J. **Differential Restriction Categories**.

Example

- The category of smooth functions defined on open subsets is a differential restriction category.
- Any Cartesian differential category is a differential restriction category where $\overline{f} = 1$, so every map is total.
- Conversly, the subcategory of maps such that $\overline{f} = 1$ in a differential restriction category is a Cartesian differential category.

The Differential Category World: It's all connected!

Hope you enjoyed it!
Thanks for listening!
Merci!



Go Habs Go!