THE FREE TANGENT CATY ON AN OBJECT WITH AFFINE CONNECTION

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1) AFFINE CONNECTIONS

Let $\mathcal{C}$ be a tangent caty, $M \in \mathcal{C}$. An <u>affine connection</u> on M is a map $K : T^2M \longrightarrow TM$ st:

- $$\begin{tikzcd}
& T^2M \arrow[dl, "Tp"'] \arrow[d, "pT"] \arrow[dr, "K"] & \\
TM \arrow[dr, "p"'] & TM \arrow[d, "p"] & TM \arrow[dl, "p"] \\
& M &
\end{tikzcd}$$ $\circledast$ is a limit

- various other axioms.

Saw yesterday: the free tang. caty on an object is <u>Weil</u>$_1$.
What's the corresponding free tang. caty $\mathbb{F}$ on an obj M w/ affine connection?
Goal: sketch an answer.

2) LOCAL COORDINATES

Since $M \in \mathbb{F}$ has an affine connection, by $\circledast$ have $T^2M \cong \overset{T_3M}{\overset{\shortparallel}{TM \times_M TM \times_M TM}}$ via Tp, pT, K. More generally, since T pres pb's of p_M's, have $T(T_kM) \cong T_{2k+1}M$. So <u>objects</u> of $\mathbb{F}$ can be taken as

$$\{T_kM : k \geq 0\} \qquad \text{with} \qquad T(T_kM) = T_{2k+1}M.$$

Note that the T_kM's are the distinct powers of TM over M:

so $\mathbb{F}$ is in fact a Lawvere theory.

NOTATION • Write gen. elements of $T_k M$ as $\vec{a} = (a_1, \dots, a_n)$ via $\pi_i : T_k M \to TM$

• Write gen. elements of $T(T_k M)$ as $(u, \vec{a}, \vec{b})$ via
- $Tp \circ T\pi_1$
- $pT \cdot T\pi_i$
- $k \cdot T\pi_i$.

Goal: figure out what things have to be in our Lawvere theory in terms of these "local coords".

3) THE BASICS

Since we have $+_M$, 0_M, our Lawvere theory includes the theory of comm. monoids. Using this, can dispatch $p, +, 0, \ell$:

$$p_{T_k M} : T(T_k M) \to T_k M \qquad (u, \vec{a}, \vec{b}) \mapsto \vec{a}$$

$$0_{T_k M} : T_k M \to T(T_k M) \qquad \vec{a} \mapsto (0, \vec{a}, \vec{0})$$

$$+_{T_k M} : T(T_k M) \times_{T_k M} T(T_k M) \to T(T_k M)$$
$$(u, \vec{a}, \vec{b}),\ (v, \vec{a}, \vec{c}) \mapsto (u+v, \vec{a}, \vec{b}+\vec{c})$$

$$\ell : T(T_k M) \to T^2(T_k M)$$
$$(u, \vec{a}, \vec{b}) \mapsto (0, (0, \vec{a}, \vec{0}), (u, \vec{0}, \vec{b}))$$

4) TORSION

Prop The symm. map $c_M : T^2M \longrightarrow T^2M$ given by

$$(u,a,b) \longmapsto (a,u,\, b+\underbrace{T(a,u)}_{\text{"torsion"}})$$

for some <u>bilinear</u> $T_2M \longrightarrow TM$ st $T(a,b)+T(b,a)=0$.

Proof First, $T^2M \xrightarrow{Tp} TM$ and $T^2M \xrightarrow{pT} TM$ are comm. monoids over TM via

$$(u,a,b) +_{pT} (v,a,c) = (u+v,\, a,\, b+c)$$
$$(u,a,b) +_{Tp} (u,c,d) = (u,\, a+c,\, b+d).$$

Clearly c must be of form $c(u,a,b) = (a,u,\varphi(u,a,b))$.
Since c a map of comm. monoids $pT \to Tp$ and $Tp \to pT$, have

$$\varphi(u,0,0) = 0 = \varphi(0,a,0)$$
$$\left.\begin{aligned}\varphi(u+v,a,b+c) &= \varphi(u,a,b)+\varphi(v,a,c)\\ \varphi(u,a+c,b+d) &= \varphi(u,a,b)+\varphi(u,c,d)\end{aligned}\right\} \quad (*)$$

So $\varphi(u,a,b) = \varphi(u,a,0) + \varphi(0,a,b)$
$$= \varphi(u,a,0) + \underbrace{\varphi(0,a,0)}_{0} + \varphi(0,0,b)$$

But $c\ell = \ell$ implies $\varphi(0,0,b) = b$. So writing $T(a,u) = \varphi(u,a,0)$, get

$$\varphi(u,a,b) = T(a,u) + b.$$

Bilinearity of T comes from $(*)$; $T(a,u)+T(u,a)=0$ since $c^2 = 1$. $\square$

4) CURVATURE

Similar argument shows:

PROP $c_{TM} : T^3M \longrightarrow T^3M$ given by

$$(v, (u,a,b), (u',a',b')) \longmapsto (u, (v,a,a'), (v'+T(u,v), b, b'+R(u,v,a)))$$

for some trilinear $R : T_3M \to TM$ st $R(a,b,c)+R(a,c,b)=0$.
"curvature".

In fact c_{T_kM} given by a similar formula.

5) COVARIANT DERIVATIVE

Leaves only action of T on maps. Note everything so far has only involved sums of reindexings of multilinear maps. So enough to say how T acts on these.

PROP If $f: T_kM \longrightarrow TM$ is multilinear, then

$$Tf : (u, \vec{a}, \vec{b}) \longmapsto (u, f(\vec{a}), \nabla f(u,\vec{a}) + \sum_{i=1}^{k} f(\vec{a}[b_i/a_i]))$$

for some $(k+1)$-linear $\nabla f : T_{k+1}M \to TM$.
covariant deriv of f.

Using this, can also describe how T acts on sums of reindexings of multilinear maps.

6) Axioms

Functorality of T says:

(1) $\nabla(\mathrm{id})(u, \vec{a}) = 0$

(2) $\nabla(g \circ (f_1, \dots, f_k)) = (\nabla g) \circ (f_1, \dots, f_k) + g \circ (\nabla f_1, f_2, \dots, f_k) + \dots + g \circ (f_1, \dots, f_{k-1}, \nabla f_k)$

chain rule

Mac Lane hexagon for c_M:

(3) $\mathfrak{S}(R(x,y,z)) = \mathfrak{S}(\nabla T(x,y,z) + T(T(x,y),z))$

$\mathfrak{S}$ = cyclic sum

Bianchi identities

Mac Lane hexagon for c_{TM}:

(4) $\mathfrak{S}(\nabla R(w,x,y,z)) + \mathfrak{S}(R(T(w,x),y,z)) = 0$

Nat. of c:

(5) $R(u,v,f(\vec{x})) + \nabla^2 f(v,u,\vec{x}) = \sum_{i=1}^{k} f(\vec{x}[R(u,v,x_i)/x_i]) + \nabla f(T(u,v),\vec{x}) + \nabla^2 f(u,v,\vec{x})$

for all k-multilinear f

Ricci identity.

7) Solution (hopefully)

Defn An operad w/ connection is a symm. CMan-operad $\mathcal{O}$ with:

- an equivariant operation $\nabla : \mathcal{O}(n) \to \mathcal{O}(n+1)$
- elements $T \in \mathcal{O}(2)$, $R \in \mathcal{O}(3)$ suitably antisymm.
- satisfying (1) — (5).

CONJ There's an adjunction

$$\text{Tang Aff Conn} \underset{\text{End}}{\overset{\text{Free}}{\underset{\perp}{\leftrightarrows}}} \text{OWC}$$

tangent cats w/ an object w/ affine connection

operads w/ connection

where $\text{End}(\mathcal{C}, M, K)$ is the operad of multilinear maps

$$\text{End}(\mathcal{C}, M, K)(n) = \{T_n M \to TM\}$$

and where $\text{Free}(\mathcal{O})$ is the Lawvere theory of $\mathcal{O}$-algebras.

If this holds, then the free tangent cat w/ affine connection is the Lawvere theory of algs over the initial OWC.