A groupoid of permutation trees

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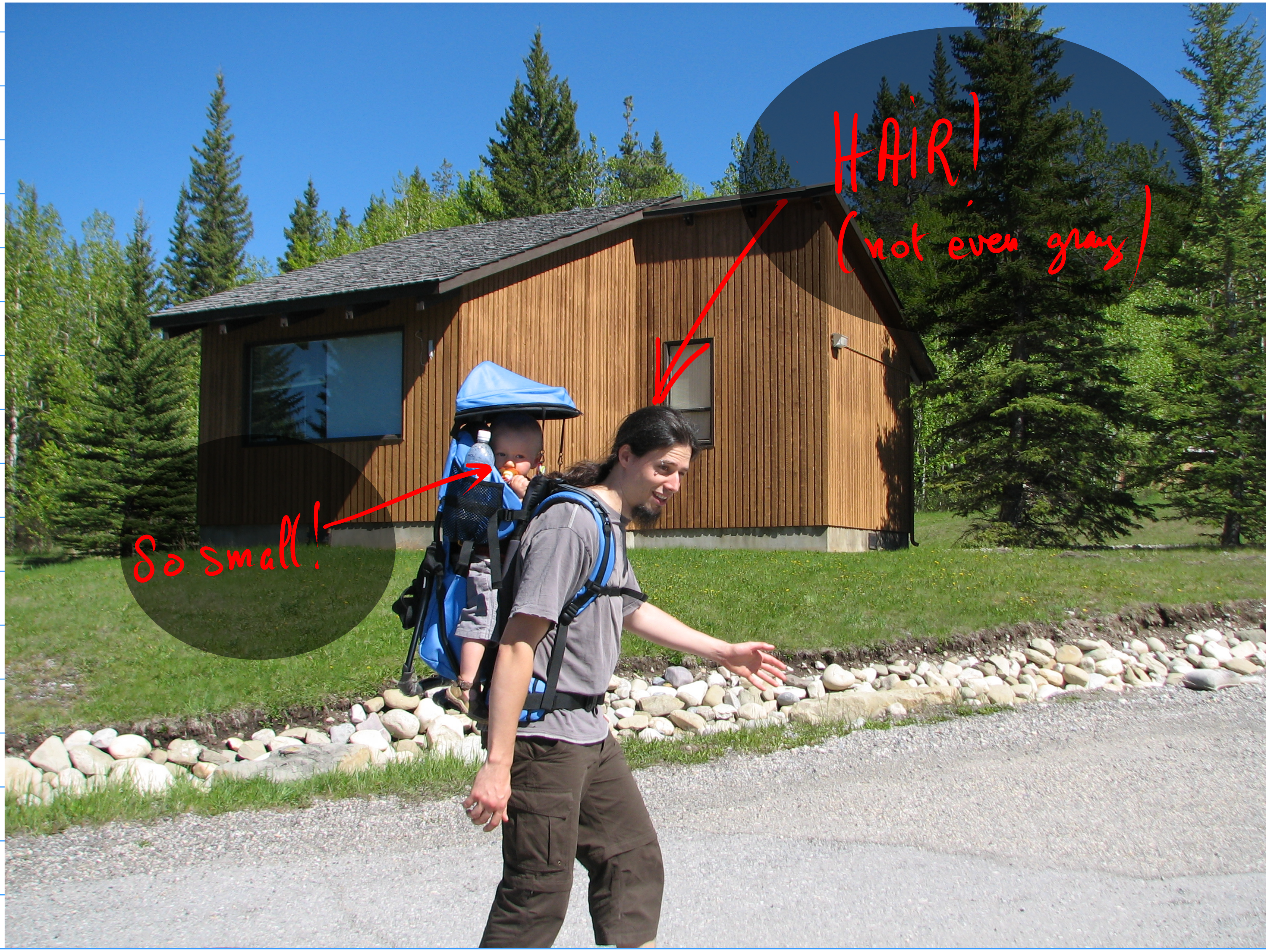
Based on https://arxiv.org/abs/2008.02665 w/ Federico Olimpieri (LIPN, Paris)

Prelude: FMCS 2010 (Kananaskis)



0: (Left) action of permutations on lists

Given a set A: $\quad \mathfrak{S}_n$ acts on A^n

$$A^n \ni \vec{a} = (a_1, \dots, a_n)$$

$$[\sigma]\,\vec{a} := \vec{a}' = (a'_1, \dots, \overset{a_1}{\overset{\|}{a'_{\sigma(1)}}}, \dots, a'_n)$$

with $\quad a'_i := a_{\sigma^{-1}(i)}$

Indeed:

$$[1_n]\,\vec{a} = \vec{a}$$

$$[\tau][\sigma]\,\vec{a} = [\tau\sigma]\,\vec{a}$$

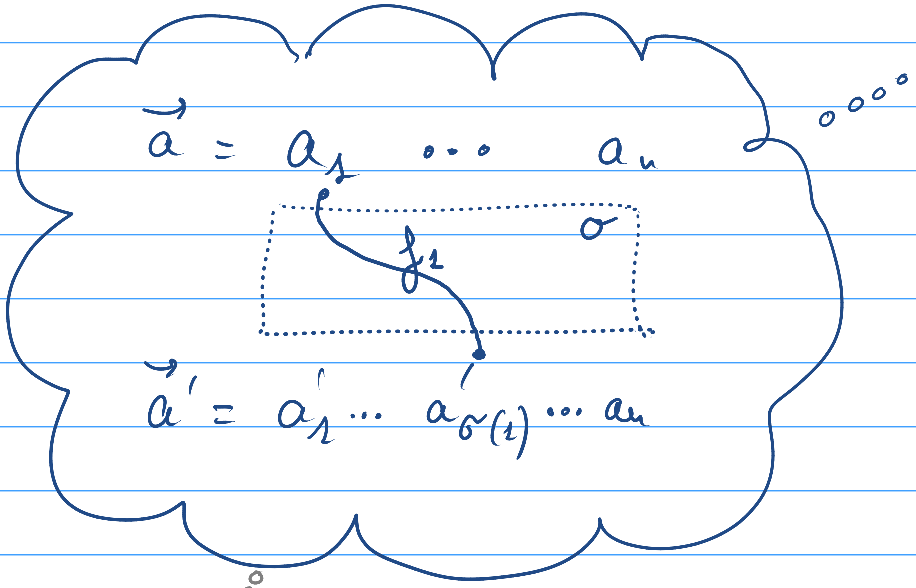
Given a category A:

Free sym. strict mon. cat. $\mathbb{P}(A)$:

objects: $A^* = \bigcup_n A^n$

morphisms $\vec{a} \longrightarrow \vec{a}'$

$(\sigma, f_1, \dots, f_n)$ with $f_i : a_i \to a'_{\sigma(i)}$



Equivalently:

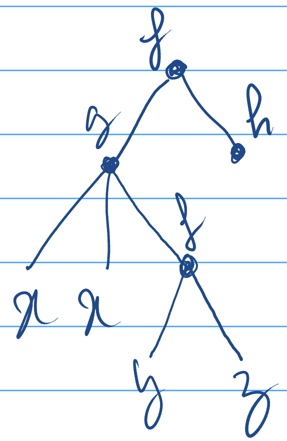
$(\sigma, \vec{f}) : \vec{a} \to \vec{a}'$ iff $\vec{f} : \vec{a} \to [\sigma^{-1}]\vec{a}'$

in $\vec{A} := \sum_n A^n$

these slides
include little clouds
à la Robin Cockett

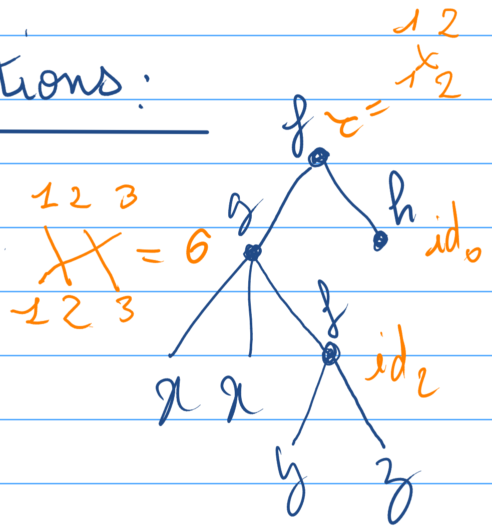
1. Permutation trees

Trees ...: $f, g, h, \ldots \in F \qquad x, y, z, \ldots \in V$

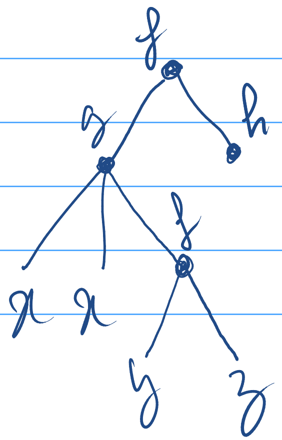


$$= f\big(g(x, x, f(y,z)),\ h()\big)$$

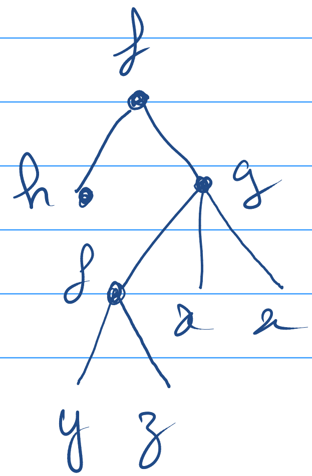
... with permutations:



$$= f^{\tau}\big(g^{6}(x, x, f^{id}(y,z)),\ h^{id}()\big)$$



$\longrightarrow$

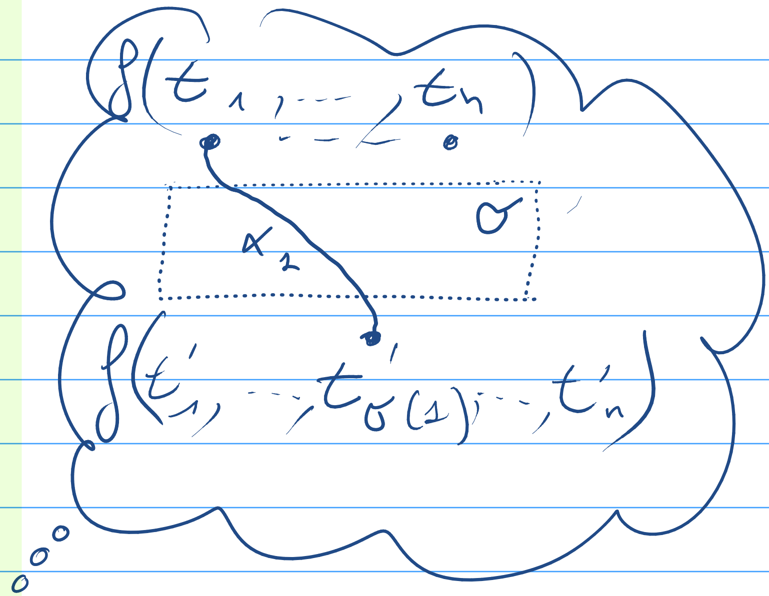


The groupoid $\mathbb{G}$:

- objects : $T(F, V)$
- morphisms : $\alpha : t \to t'$

$$\frac{}{x : x \to x}$$

$$\frac{\alpha_1 : t_1 \to t'_{\sigma(1)} \quad \cdots \quad \alpha_n : t_n \to t'_{\sigma(n)}}{f^{\sigma}(\alpha_1, \dots, \alpha_n) : f(t_1, \dots, t_n) \to f(t'_1, \dots, t'_n)}$$



Remark: $\mathbb{G} = V + F \times \mathbb{P}(\mathbb{G})$

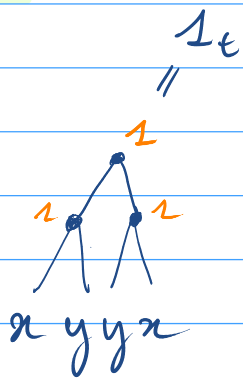
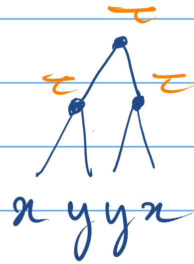
Multiplicity degree

$$m(t) := \# \mathbb{G}(t) \quad \dots \quad \mathbb{G}(t) := \mathbb{G}(t,t)$$

$$m(\vec{t}) := \# P(\mathbb{G})(\vec{t})$$

Example:

 $= t$

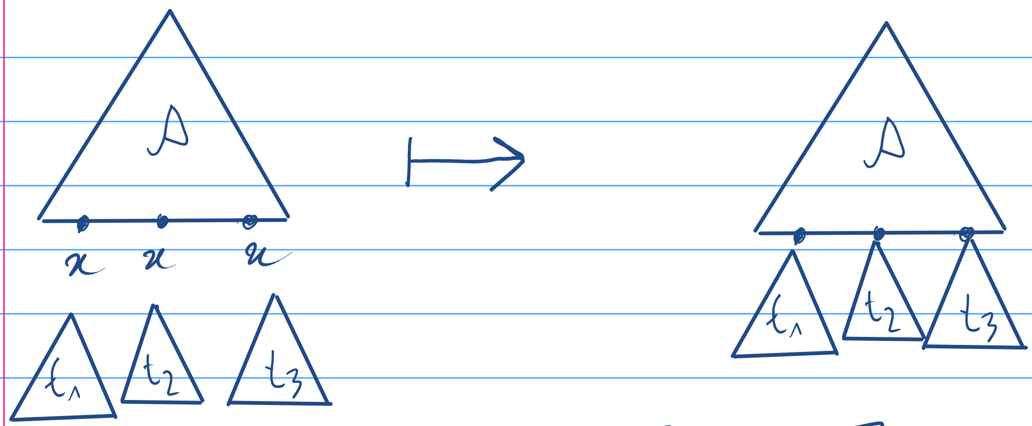
$\mathbb{G}(t) = \{$  ,  $\}$ $\Rightarrow m(t) = 2$

Def: $t \sim t'$ if $\exists \alpha : t \to t'$

Lemma:

$$\# \mathbb{G}(t,t') = \begin{cases} m(t) & \text{if } t \sim t' \\ 0 & \text{otherwise} \end{cases}$$

2. Substitution and permutations



$$=: s[t_1 t_2 t_3 / x]$$

Substitution: $s[\vec{t}/x]$ (ass. $n_x(s) = |\vec{t}|$)

$$x[(t_1)/x] := t_1$$

$$y[()/x] := y$$

$$f(s_1,\dots,s_n)[\vec{t}/x] := f(s_1[\vec{t_1}/x], \dots, s_n[\vec{t_n}/x])$$

assuming $\vec{t} = \vec{t_1} \dots \vec{t_n}$ with $n_x(s_i) = |\vec{t_i}|$

much like the operad of (unlabelled) trees

Goal

Relate $\quad m\left(s[\vec{t}/x]\right) \quad$ or $\quad \mathbb{G}\left(s[\vec{t}/x]\right)$

with $\quad m(s) \quad$ and $\quad m(\vec{t})$

or $\quad \mathbb{G}(s) \quad$ or $\quad \mathbb{P}(\mathbb{G})(\vec{t})$

Substitution of morphisms:

Def: $\alpha[\vec{\beta}/x]$ $\quad \left(\text{ass. } |\vec{\beta}| = n_x(\alpha) \right)$

$$x[(\beta_1)/x] = \beta_1 \qquad\qquad y[()/x] = y$$

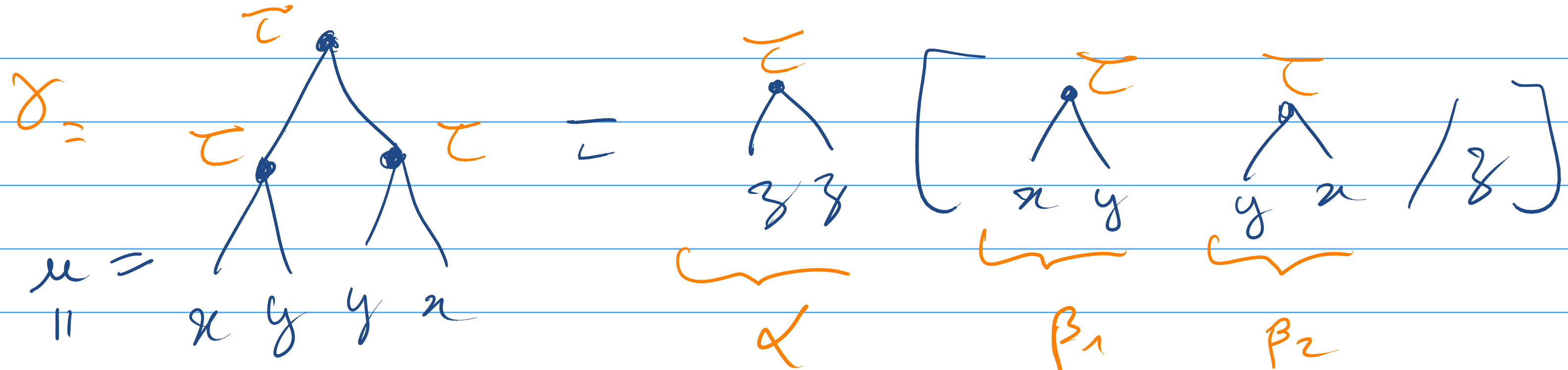
$$f^\sigma(\vec{\alpha})[\vec{\beta}/x] := f^\sigma\left(\alpha_1[\vec{\beta}_1/x], \ldots, \alpha_n[\vec{\beta}_n/x]\right)$$

(assuming $\vec{\beta} = \ldots$)

Let: $\begin{cases} \alpha : s \longrightarrow s' \\ \vec{\beta} : \vec{t} \longrightarrow \vec{t}' \ \ldots\ldots \text{ in } \vec{\mathbb{G}} \end{cases}$

Then: $\alpha[\vec{\beta}/x] : s[\vec{t}/x] \longrightarrow ?$ $\quad s'[\vec{t}'/x]$?

Example



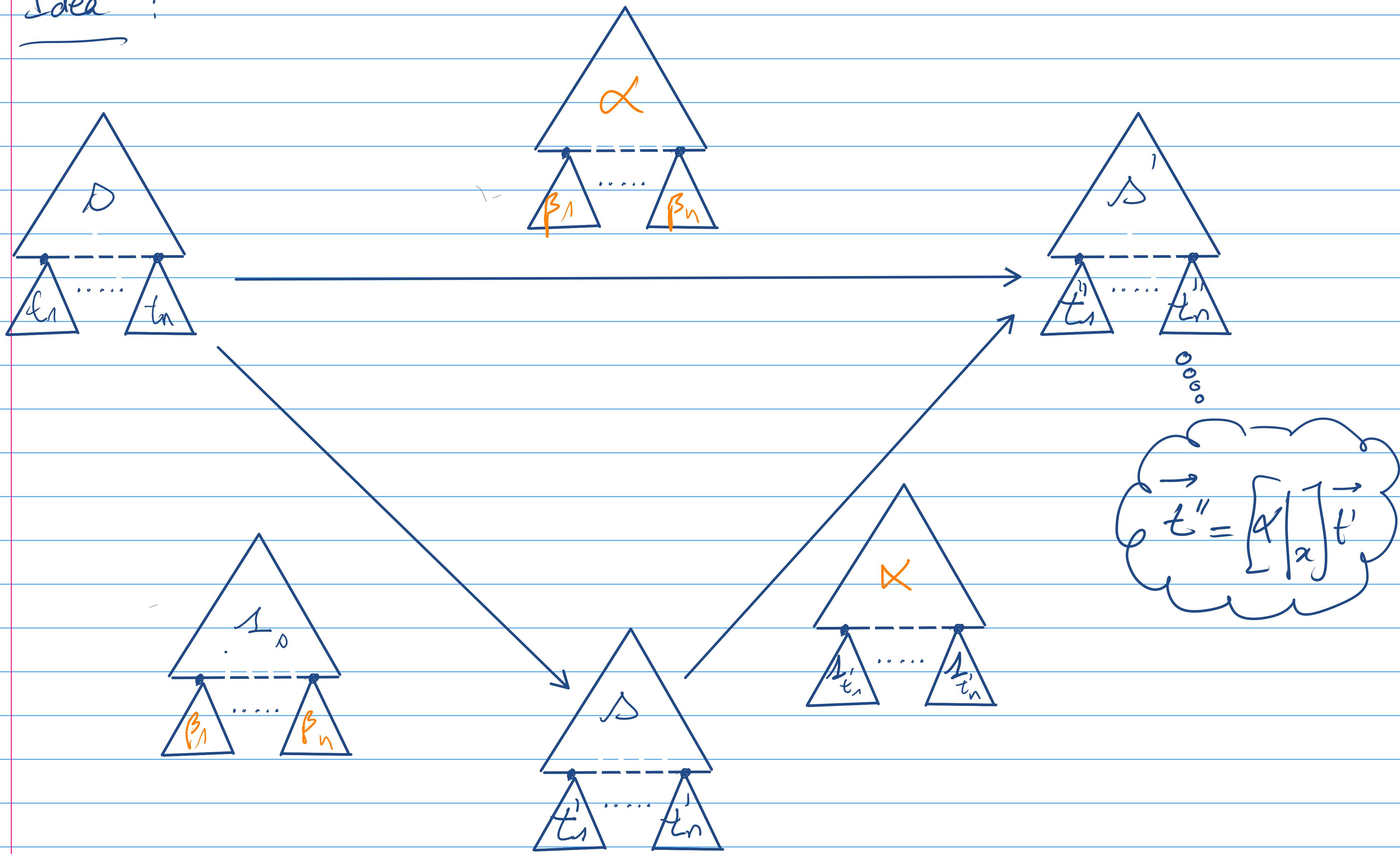
$s[t_1 t_2/x]$

$$u = s[\,t_1\; t_2/x\,]$$

$$\downarrow \qquad\qquad \downarrow^{\alpha} \quad \downarrow^{\beta_1} \quad \downarrow^{\beta_2}$$

$$u \neq s[\,t_2\; t_1/x\,]$$

Idea :



α
β_1
β_n
s
t_1
t_n
s'
t''_1
t''_n
1_s
β_1
β_n
s
t'_1
t'_n
α
$1_{t'_1}$
$1_{t'_n}$
$\vec{t}'' = [\alpha/_x]\vec{t}'$

Permutation induced by a morphism

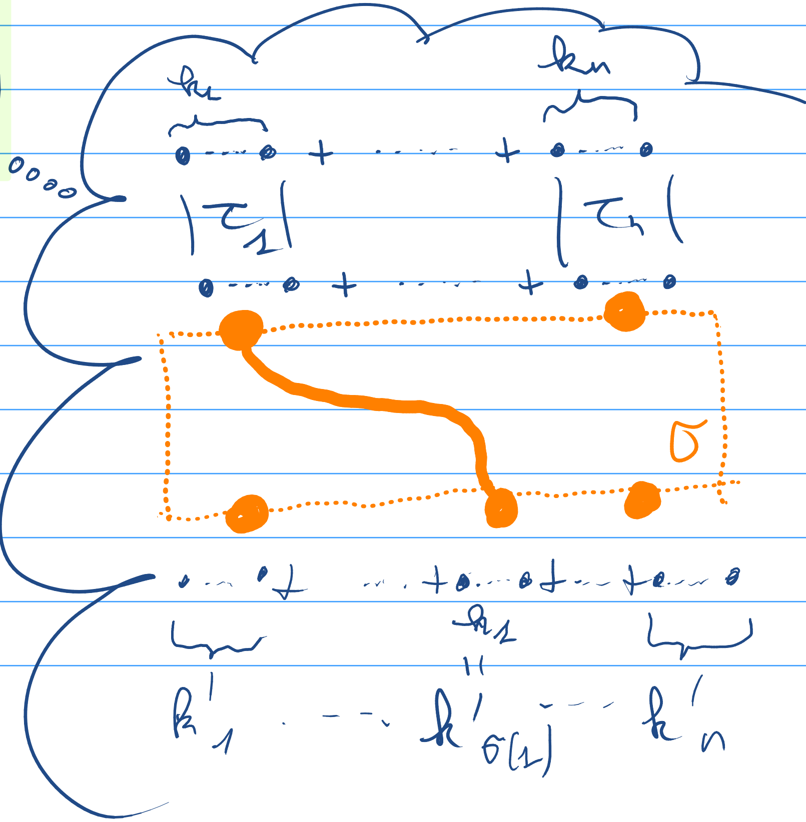
$\alpha : s \to s' \rightsquigarrow \alpha|_x \in \mathbb{S}_{n_x(s)}$ (functorially)

Def: $x|_x = 1_1 \qquad y|_x = 1_0 \qquad f^{\sigma}(\vec{\alpha})\Big|_x = \sigma \bullet (\alpha_1|_x, \dots, \alpha_n|_x)$

Multiplexing:

$$\sigma \bullet [\tau_1, \dots, \tau_n] := \sigma_{k_1 \dots k_n} \circ (\tau_1 \otimes \dots \otimes \tau_n)$$

$$\begin{array}{cccc} \cap & \cap & \cap & \cap \\ \mathbb{S}_n & \mathbb{S}_{k_1} & \mathbb{S}_{k_n} & \mathbb{S}_{\sum_{i=1}^{n} k_i} \end{array}$$



(Operad on $(\mathbb{S}_n)_{n \in \mathbb{N}}$)

Target of a substitution

If $\alpha : s \rightsquigarrow s'$ and $\vec{\beta} : \vec{t} \to \vec{t'}$

then $\alpha[\vec{\beta}/x] : s[\vec{t}/x] \longrightarrow s'[[\alpha]_x \vec{t'}/x]$

We obtain:

$$\mathbb{G}(s,s') \quad \vec{\mathbb{G}}(\vec{t},\vec{t'}) \quad \mathbb{G}(s[\vec{t}/x], s'[[\alpha]_x \vec{t'}/x])$$

$$(\alpha \in \mathbb{G}(s,s'),\ \vec{\beta} \in \vec{\mathbb{G}}(\vec{t},\vec{t'})) \longmapsto \alpha[\vec{\beta}/x] \in \mathbb{G}(s[\vec{t}/x], s'[[\alpha]_x \vec{t'}/x])$$

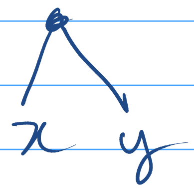
clearly injective
($s, \vec{t}$ being fixed)

Hence:

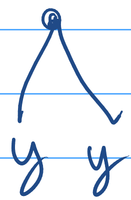
$$-[-/x] : \sum_{\alpha \in G(s)} \vec{G}\left(\vec{t}, [\alpha|_x^{-1}]\vec{t}\right) \hookrightarrow G\left(s[\vec{t}/x]\right)$$

⚠ no functoriality is claimed

Surjectivity?

$s =$ 

$\vec{t} = (\ y\ )$

$s[\vec{t}/x] =$ 

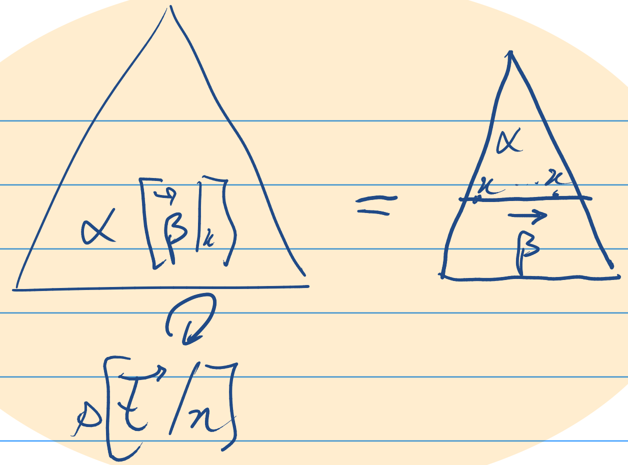
$G(s) = \{1_s\}$

$\vec{G}(\vec{t}) = \{(1_t)\}$

$G(s[\vec{t}/x]) = \{f^1(y,y), f^2(y,y)\}$

No!

3. The uniform case

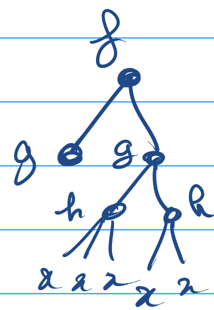


Coherence:

$$\frac{}{x \asymp x} \qquad\qquad \frac{t_i \asymp t_j \quad 1 \le i,j \le n+k}{f(t_1,\dots,t_n) \asymp f(t_{n+1},\dots,t_{n+k})}$$

Def: s is uniform if $s \asymp s$

Ex:



...

very restrictive at this stage!!!

Surjectivity:

Lemma: If $\gamma : s[\vec{t}/x] \longrightarrow s'[\vec{t}'/x]$ with $s \supset s$

then $\gamma = \alpha[\vec{\beta}/x]$ with

$\alpha : s \to s'$ and $\vec{\beta} : \vec{t} \longrightarrow [\alpha|^{-1}_x]\,\vec{t}'$

Hence: If s is uniform

$$-[-/x] : \sum_{\alpha \in \mathbb{G}(s)} \vec{\mathbb{G}}(\vec{t}, [\alpha|^{-1}_x]\,\vec{t}) \xrightarrow{\sim} \mathbb{G}(s[\vec{t}/x])$$

Fast forward ...

Up to some group theory magic, we get a bijection:

$$\mathbb{G}(s) \times \mathbb{P}(\mathbb{G})(\vec{t}) \simeq \mathbb{H}_x(s, \vec{t}) \times \mathbb{G}(s[\vec{t}/x])$$

where $\mathbb{H}_x(s, \vec{t}) := \{ \sigma \in \mathfrak{S}_n \;;\; s[\vec{t}/x] \sim s[[\sigma]\vec{t}/x] \}$

(always assuming s uniform)

4. Why?

$\Lambda \ni M, N, \ldots ::= x \mid \lambda x M \mid M N$

$\Delta \ni s, t, \ldots ::= x \mid \lambda x s \mid \langle s \rangle [t_1, \ldots t_n]$

Taylor expansion of λ-terms:

$$\mathcal{T} : \Lambda \longrightarrow \mathbb{Q}^{\Delta}$$

$$x \longmapsto x$$

$$\lambda x M \longmapsto \lambda_x \mathcal{T}(M) = \sum_{s \in \Delta} \mathcal{T}(M)_s \cdot \lambda x s$$

$$M N \longmapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} \langle \mathcal{T}(M) \rangle \left[\mathcal{T}(N)\right]^n$$

$$= \sum_{n \in \mathbb{N}} \sum_{s, t_1, \ldots, t_n \in \Delta} \frac{\mathcal{T}(M)_s \, \mathcal{T}(N)_{t_1} \cdots \mathcal{T}(N)_{t_n}}{n!} \cdot \langle s \rangle [t_1, \ldots t_n]$$

Th (Ehrhard – Regnier):

$$\mathcal{T}(M)_s = \begin{cases} m(s)^{-1} \\ 0 \end{cases}$$

oooooo

permutations on $[t_1, \ldots t_n]$ only

Dynamics:

$$(\lambda x M) N \to M[N/x]$$

$$\langle \lambda x s\rangle [t_1,\dots,t_n] \to \partial_x s \cdot [t_1,\dots,t_n] = \begin{cases} \sum_{\sigma\in \mathfrak{S}_n} s[[\sigma]\vec{t}/x] \\ 0 \end{cases}$$

Th (Ehrhard-Regnier):

$$NF(\mathcal{T}(M)) = \mathcal{T}(BT(M))$$

Relies on: **Lemma:**

$$\left(\partial_x s \cdot [t_1,\dots,t_n]\right)_u = \frac{m(s)\, m(\vec{t})}{m(u)}$$

$$\quad /\!/ \quad \#\{\sigma \in \mathfrak{S}_n \,;\, s[[\sigma]\vec{t}/x] = u\}$$

Boils down to:

$$\mathbb{G}(s) \times \mathbb{P}(\mathbb{G})(\vec{t}) \simeq H_x(s,\vec{t}) \times \mathbb{G}(s[\vec{t}/x])$$

which holds thanks to the uniformity of Taylor approximants

coherence condition
only between elements of bags

Conclusion

- This was a lot of fun !
- Connected to T.A.O.'s approach to gen. species
  (but not quite the same)
- Might lead to a "syntactic" presentation of reflexive object
  in the cart. closed bicat. of gen. species
- Would be nice to find a more categorical account
  of the bijection.

Merci

4. Application to the Taylor expansion of λ-terms

$$\mathcal{T} : \Lambda \longrightarrow \mathbb{Q}^{\Delta}$$

$$M \longmapsto \mathcal{T}(M) = \sum_{s \in \Delta} M_s \cdot s$$

$$x \longmapsto x$$

$$\lambda x\, M \longmapsto \lambda x\, \mathcal{T}(M) = \sum_{\Delta} M_s \cdot \lambda x\, s$$

$$M\, N \longmapsto \sum_{n \in \mathbb{N}} \frac{1}{n!} \underbrace{\langle \mathcal{T}(M) \rangle\; \mathcal{T}(N)^n}_{D_0^n M \cdot N^n} = \sum_{n} \frac{1}{n!} \sum_{s, t_1 \dots t_n} \Big(M_s \cdot \prod_i N_{t_i} \Big) \langle s \rangle [t_1, \dots, t_n]$$

Where: $\Delta \ni s, t ::= x \mid \lambda x\, s \mid \langle s \rangle [t_1, \dots, t_n]$

Th: $NF(\mathcal{T}(M)) = \mathcal{T}(BT(M))$

Lemma: $$M_s = \begin{cases} \frac{1}{m(s)} \\ 0 \end{cases}$$

Rigid resource λ-calculus

$$D \ni s, t ::= x \mid \lambda x s \mid \langle s \rangle (t_1, \ldots, t_n)$$

$$D \longrightarrow \Delta$$

$$s \longmapsto \|s\|$$

$\|x\| = x$ $\quad \|\lambda x s\| = \lambda x \|s\|$ $\quad \|\langle s\rangle(t_1, \ldots, t_n)\| = \langle \|s\| \rangle [\|t_1\|, \ldots, \|t_n\|]$

$$G : \quad f, g ::= 1_x \mid \lambda x f \mid \langle f \rangle (\sigma, g_2, \ldots, g_n)$$

Lemma: if $s \in D$ $\quad m(\|s\|) := m(s) := \# G(s)$

Dynamics : $\| (\lambda x s)\, \vec{t} \| \to \sum_{\sigma \in \mathfrak{S}_n} \| s[[\sigma]\vec{t}/x] \|$

Let $u \in \{ s[[\sigma]\vec{t}/x] \,;\, \sigma \in \mathfrak{S}_n \}$

We want

Lemma: $\dfrac{1}{m(u)} = \dfrac{1}{m(s)\, m(\vec{t})} \times \# \{ \sigma \in \mathfrak{S}_n \text{ s.t. } s[[\sigma]\vec{t}/x] \sim u \}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \; \mathbb{H}_x(s, \vec{t}) \text{ if } u = s[\vec{t}/x]$

Holds as soon as $s \frown s$:

Def (coherence) : $\quad x \frown x \qquad \dfrac{s \frown s'}{\lambda x s \frown \lambda x s'} \qquad \dfrac{s \frown s' \qquad t_i \frown t_j \;\; (1 \leq i, j \leq n+k)}{\langle s \rangle (t_1 \ldots t_n) \frown \langle s' \rangle (t_{n+1} \ldots t_{n+k})}$

Rem: support $(\mathcal{T}(M))$ is a clique

FIN