# Barycentric calculus, and the log-exp bijection

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Presented (online) at
"Tangent Categories and their Applications"
BIRS, Banff, June 2021

Thank you to the organizers !

Message:

Affine geometry on an infinitesimal scale
is possible in any good space

Need to explain the three key phrases. The first is standard: affine geometry is the geometry whose calculational aspect is barycentric calculus: barycentric calculus = the calculus of linear combinations where the sum of the coefficients is 1, e.g. formation of parallelograms.

What is a good space? Wrong question! It is an object in a good category of spaces.

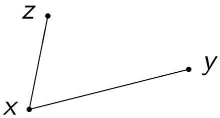
Examples (good in different ways - the word "good" has many meanings): e.g. a tangent category, or a topos model of SDG.

Also: the category $\mathcal{A}^{op}$ of affine schemes

It is in the latter that I shall present a notion of infinitesimal scale, by defining a reflexive symmetric relation $f \sim_1 g$ on each of its hom sets.

The dual of the category of affine $k$-schemes is the category of commutative $k$-algebras, and most of the following deals with that category. For those of you who attended the talks of Finster, Joyal (and others), this is like the relationship between the category of toposes (the geometric aspect), and the category of logoses (the algebraic aspect).

# Barycentric calculus







$$1 \cdot y + (-1) \cdot x + 1 \cdot z$$

A linear combination where the sum of the coefficients is 1, also called an affine combination.

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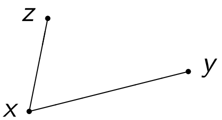


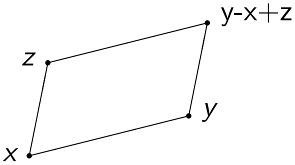


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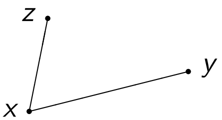


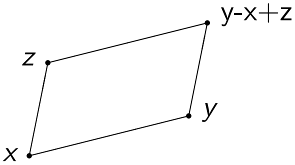


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Similarly, given two points,





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$$(1 - t) \cdot x + t \cdot y$$

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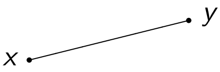


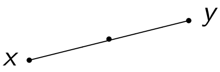


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($\mathcal{A} =$ the category of commutative $k$-algebras)

2. Some affine combinations in $\hom_{\mathcal{A}}(B, C)$

3. Geometric meaning in $\mathcal{A}^{op} =$ the category of affine schemes over $k$

"Much like the theory of affine schemes and commutative rings, the theory of (higher) topoi leads a dual life: one algebraic and one geometric. . ." cf. Finster's abstract.

4. log: from neighbour pairs to tangent vectors

# 1. (First order) neighbours

Neighbours in the category $\mathcal{A}$ of commutative $k$-algebras:

For each $B$ and $C$ in $\mathcal{A}$, and $f$ and $g$ parallel maps:

$$B \underset{g}{\overset{f}{\rightrightarrows}} C,$$

we say that they are (1st order) neighbours if for all $x \in B$:

$$f \sim_1 g :$$

$$(f(x) - g(x))^2 = 0$$

It is a reflexive and symmetric relation. It is not transitive.

Slightly stronger:

for all $x, y \in B$

$$(f(x) - g(x)) \cdot (f(y) - g(y)) = 0$$

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So $f \sim_1 g$, is defined by validity, for all $x$ and $y$ in the domain $B$, of

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This relation in $\hom_{\mathcal{A}}(B, C)$ is equivalent to

$$f(x) \cdot f(y) + g(x) \cdot g(y) = f(x) \cdot g(y) + g(x) \cdot f(y)$$

No minus occurs !
(So it works for commutative rigs as well)
The value of $f$ and $g$ on a pair of elements in $B$ gives a $2 \times 2$ matrix with entries from $C$

| | $x$ | $y$ |
|---|---|---|
| $f$ | $f(x)$ | $f(y)$ |
| $g$ | $g(x)$ | $g(y)$ |

It is stable under precompostion by any map $A \to B$ and under postcompostion by any map $C \to D$.

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In particular: a 1-neighbour of the identity map of $B$ is of the form: identity map of $B$ plus a derivation $B \to B$.
Or: first order deformation of the identity map of $\overline{B}$ = vector field on $\overline{B}$ = derivation on $B$.

We may consider not only neighbours *pairs* $f, g$ of $k$-algebra maps $B \to C$, but for an $n$-tuple of mutual neighbouring $k$-algebra maps $B \to C$,

$$B \underset{f_n}{\overset{f_1}{\underset{\vdots}{\overset{\longrightarrow}{\longrightarrow}}}} C$$

For $x, y \in B$, we get similarly an $n \times 2$ matrix

| | $x$ | $y$ |
|---|---|---|
| $f_1$ | $f_1(x)$ | $f_1(y)$ |
| $f_2$ | $f_2(x)$ | $f_2(y)$ |
| . | . | . |
| . | . | . |
| $f_n$ | $f_n(x)$ | $f_n(y)$ |

# 2. Affine combinations of $n$-tuples of mutual neighbours

Theorem.
Let $f_1, \ldots, f_n$ be an $n$-tuple of mutual neighbour $k$-algebra maps $B \to C$, and let $t_1, \ldots, t_n$ be elements of $C$ with $t_1 + \ldots + t_n = 1$. Then the affine combination

$$\sum_{i=1}^{n} t_i \cdot f_i : B \to C$$

is a $k$-algebra map. The construction is natural in $B$ and in $C$.

Proof for $n = 2$. Let $x$ and $y$ in $B$. Multiplication in $B$ denoted $\bullet$:

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

$$s + t = 1 \text{ in } C$$

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$$(s \cdot f + t \cdot g)(x \bullet y) =$$

$$(s \cdot f)(x \bullet y) + (t \cdot g)(x \bullet y) =$$

$$s \cdot f(x) \cdot f(y) + t \cdot g(x) \cdot g(y)$$

since both $f$ and $g$ preserve multiplication.
Two terms!

$$(s \cdot f + t \cdot g)(x) \cdot (s \cdot f + t \cdot g)(y)$$

multiply out in $C$, get four terms

$$s^2 \cdot f(x) \cdot f(y) + s \cdot t \cdot f(x) \cdot g(y) + t \cdot s \cdot g(x) \cdot f(y) + t^2 \cdot g(x) \cdot g(y)$$

So compare with the previous poor little two-term expression:

$$s \cdot f(x) \cdot f(y) + t \cdot g(x) \cdot g(y)$$

Multiply it by $s + t = 1$! Get four terms

$$s^2 \cdot f(x) \cdot f(y) + t \cdot s \cdot f(x) \cdot f(y) + s \cdot t \cdot g(x) \cdot g(y) + t^2 \cdot g(x) \cdot g(y)$$

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The leftmost red term matches the leftmost blue term.
The rightmost red term matches the rightmost blue term.
The remaining (middle) terms are ($s \cdot t$ times) respectively

$$f(x) \cdot f(y) + g(x) \cdot g(y)$$

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and they are equal by the "cross equation" defining $f \sim_1 g$. QED

$$s^2 \cdot f(x) \cdot f(y) + t \cdot s \cdot f(x) \cdot f(y) + s \cdot t \cdot g(x) \cdot g(y) + t^2 \cdot g(x) \cdot g(y)$$

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The theorem may be augmented with the following, which are proved in a similar elementary way:

Let $f_1, \ldots, f_n$ be an $n$-tuple of mutual neighbour $k$-algebra maps $B \to C$.

Then any two affine combinations (with coefficients from $C$) of these maps are neighbours.

Or: in the $C$-module of $k$-linear maps $B \to C$, the affine span of the $f_i$s consists of $k$-algebra maps, and they are mutual neighbours.

In particular, the barycenter of a $(p+1)$ tuple of mutual 1-neighbour maps $B \to C$ is a well defined map $B \to C$;

In particular, the barycenter of a $(p+1)$ tuple of mutual 1-neighbour maps $B \to C$ is a well defined map $B \to C$; e.g. the mid-"point" of two neigbour maps;

and equations between such affine combinations do hold, since they are derived from the $C$-module ("vector space") $\hom_k(B, C)$ (unlike infinitesimal constructions using the tangent spaces $T_p(M)$ which give a special status to the base "point" $p$).

Thus for $x \sim y$,

$$\text{midpoint of } x \text{ and } y \; = \; \text{midpoint of } y \text{ and } x$$

# 3. Geometric meaning

$\mathcal{A}$ the category of $k$-algebras
$\mathcal{A}^{op}$ the dual category.
so notation and terminology changes.
(similar to the logos vs topos duality in the presentations by Finsler and Joyal)

$$B \underset{g}{\overset{f}{\rightrightarrows}} C$$

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"$B$ a $k$-algebra" becomes "$\overline{B}$ is an (affine) scheme (over $k$)"
The reflexive symmetric relation $\sim_1$ on the set $\hom_{\mathcal{A}}(B, C)$ is also a reflexive symmetric relation $\sim_1$ on the set $\hom_{\mathcal{A}^{op}}(\overline{C}, \overline{B})$, i.e. on points of $\overline{B}$ (defined at same stage $\overline{C}$))

The initial $k$-algebra $k$ becomes the terminal affine scheme: $\overline{k} = 1$

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Maps $\overline{C} \to \overline{B}$ (i.e. scheme maps) are called "*points* of $\overline{B}$ (defined at *stage* $\overline{C}$)"; Kripke-Joyal semantics applies in $\mathcal{A}^{op}$.
Thus points of $\overline{B}$ at stage 1 ("global" points) are the same as $k$-algebra maps $B \to k$.

What about the "scalars" $t_i$, i.e. the elements of $C$ that enter in the formulation of the Main Theorem ?
$k[X]$ = the free $k$-algebra in one generator $X$; $R := \overline{k[X]}$

Elements $t$ of $C$

correspond to

$k$-algebra maps $k[X] \to C$ (namely the one with $X \mapsto t$)
= maps in $\mathcal{A}^{op}$: $\overline{C} \to R$

points of $R$ defined at stage $\overline{C}$
or: "scalars" $\in R$ defined at stage $\overline{C}$,

$\hom_k(B, C)$ is a $C$-module.
The Theorem can be expressed:
The subset $\hom_{\mathcal{A}}(B, C) \subseteq \hom_k(B, C)$ is stable under formation of affine combinations (in this $C$-module) of mutual neighbours (and the formation is natural w.r.to maps $C \to C'$)

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The structure on $\hom_{\mathcal{A}}(B, C)$ provided by the Theorem is a partially defined algebraic structure: (defined on mutually neighbouring maps): of such, affine combinations of such (over the $k$-module of linear maps $B \to C$) can be formed (natural w.r.to $\ldots$)

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Expressed in $\mathcal{A}^{op}$:
$\hom_{\mathcal{A}^{op}}(\overline{C}, \overline{B})$ carries structure of partially defined affine space over $C$ (defined on mutually neighbouring maps $\overline{C} \to \overline{B}$ (natural w.r.to ... )
or
One may form affine combinations of mutual neigbouring points of $\overline{B}$ (points at stage $\overline{C}$) with coefficients: scalars $\in R$ (at stage $\overline{C}$ (natural w.r.to ... ))

# 4. Logarithm

For $x \sim_1 y \in \overline{B}$ and $t \in R = \overline{k[X]}$, $x, y$ neighbour points of $\overline{B}$, $t$ a point of $R$, all three at stage $\overline{C}$, say, have

$$(1 - t) \cdot x + t \cdot y \in \overline{B}$$

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(natural in $\overline{C}$) Synthetically: for $x \sim_1 y$, this defines a map $R \to \overline{B}$

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The 1-jet of this map at 0 is a tangent vector at $x$ (in $\overline{B}$).

$$\log : \overline{B}_{(1)} \to T(\overline{B})$$

For good $B$, this is a bijection onto the subscheme of $(T(\overline{B}))_1$, the inverse is called exp. For $\overline{B} = R^n$:

$$(T(\overline{B}))_1 = \text{tangent vectors whose principal part is } \sim_1 0.$$

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And exp adds the prinincipal part to the base point.

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https://tildeweb.au.dk/au76680/

Thank you !

$$B \overset{f}{\underset{g}{\rightrightarrows}} C$$

Then $f \sim_1 g$ iff

$$(f, g) : B \otimes B \to C$$

factors through

$$B \otimes B \to B \otimes B/I^2$$

where $I$ is the kernel of the multiplication map

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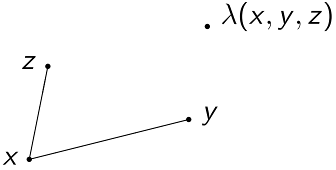
In $\mathcal{A}^{op}$, this quotent map becomes a subscheme

$$\overline{B}_1 \subset \overline{B \otimes B}$$

the “first neighbourhood of the diagonal”.
The submodule $I/I^2 \subset B \otimes B/I^2$ is the Kaehler differentials of $B$.
Interpreted in $\mathcal{A}^{op}$: the module of scalar functions $\omega : \overline{B}_1 \to R$ which vanish on the diagonal,
$\omega(x, y) \in R$ defined for $x \sim_1 y$, with $\omega(x, x) = 0$.

$\mathcal{A}$ becomes a tangent category $B \mapsto B[\epsilon]$
$\mathcal{A}^{op}$ becomes a tangent category: $B$ maps to the symmetric $B$-algebra on the Kaehler differentials of $B$.

If only $x \sim_1 y$, $x \sim_1 z$ is assumed, but not $y \sim_1 z$, the completion of the first figure into the second, is an added *structure*, namely an affine connection $\lambda$:

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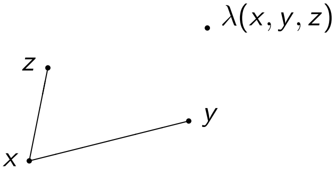


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In this case, $\lambda(x, y, z) = y - x + z$ iff $\lambda$ is symmetric in $y, z$,