Classification of SPT-phases

Yoshiko Ogata

Graduate School of Mathematical Sciences, The University of Tokyo

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Classification of SPT-Phases in 2-dimensional Fermionic systems

We derived some invariant, which is similar to the predicted one [Kapustin-Thorngren-Turzillo-Wang'15] [Brumfiel-Morgan '16], [Wang-Gu '20]
but not exactly the same. (Probably I'm missing something.)

Self-dual CAR-algebra

Definition

For a Hilbert space $\mathfrak{K}$ with a complex conjugation $\mathfrak{C}$ (i.e., anti-unitary such that $\mathfrak{C} = \mathfrak{C}^*$), self-dual-CAR-algebra $\mathfrak{A}_{\mathrm{SDC}}(\mathfrak{K}, \mathfrak{C})$ over $(\mathfrak{K}, \mathfrak{C})$ is defined as the C^*-algebra generated by $\{B(f) \mid f \in \mathfrak{K}\}$ such that

$$\mathfrak{K} \ni f \mapsto B(f), \quad \textit{linear}$$
$$\{B(f), B(g)\} = \langle f, g \rangle \mathbb{1},$$
$$B(f)^* = B(\mathfrak{C}f), \quad f, g \in \mathfrak{K}.$$

For a unitary u on $\mathfrak{K}$ with $u\mathfrak{C} = \mathfrak{C}u$, there exists an automorphism Ξ_u on $\mathfrak{A}_{\mathrm{SDC}}(\mathfrak{K}, \mathfrak{C})$ such that $\Xi_u(B(f)) = B(uf)$ for all $f \in \mathfrak{K}$. $\Theta_{\mathfrak{K}} := \Xi_{-\mathbb{1}}$ defines a grading on $\mathfrak{A}_{\mathrm{SDC}}(\mathfrak{K}, \mathfrak{C})$.

A projection p with $p + \mathfrak{C}p\mathfrak{C} = \mathbb{1}$, is called a basis projection.

For a basis projection p, there exists a unique state ω_p on $\mathfrak{A}_{\mathrm{SDC}}(\mathfrak{K}, \mathfrak{C})$ such that $\omega_p(B(pf)B(pf)^*) = 0$ for all $f \in \mathfrak{K}$. (Fock state)

SPT in 2-d Fermionic systems

Let $d \in 2\mathbb{N}$. We consider $\mathcal{A} := \mathfrak{A}_{\mathrm{SDC}}(l^2(\mathbb{Z}^2) \otimes \mathbb{C}^d, \mathfrak{C})$ with $\mathfrak{C}$ complex conjugation with respect to the standard basis.

Let G be a finite group, U_g its unitary representation on $\mathbb{C}^d$ commuting with the complex conjugation and $\beta_g := \Xi_{\mathbb{1} \otimes U_g}$ an on-site action of G.

We consider the set of β_g-invariant even interactions with a unique gapped ground state which can smoothly be deformed to trivial on-site interactions without closing the gap.

We say such two interactions are equivalent if they can be smoothly deformed into each other without closing the gap nor breaking the symmetry.

What we want to do is to derive an invariant of the classification.

SPT in 2-d Fermionic systems

Using Automorphic equivalence [Hastings-Wen '04, Bachmann et.al. '12 Nachtergaele et.al. '19, Moon-O '20], the problem is reduced as follows.

We denote by $\mathrm{QAut}(\mathcal{A})$ the set of all automorphisms on $\mathcal{A}$ given by (possibly time-dependent) even interactions.

$\mathrm{QAut}(\mathcal{A})$: automorphisms given by time-dependent interactions
Let $\Phi : [0,1] \ni t \to \Phi_t = (\Phi(X;t))$ be a continuous path of even interactions. We then define the path of local Hamiltonians $(H_{\Phi_t})_\Lambda := \sum_{X \subset \Lambda} \Phi(X;t)$ for each finite subset Λ of $\mathbb{Z}^2$ and consider the solution $\alpha_{\Phi,t,\Lambda}(A)$ of the differential equation

$$\frac{d}{dt}\alpha_{\Phi,t,\Lambda}(A) = i\left[(H_{\Phi_t})_\Lambda, \alpha_{\Phi,t,\Lambda}(A)\right], \quad \alpha_{\Phi,0,\Lambda}(A) = A.$$

If Φ is local enough, the limit $\alpha_{\Phi,t}(A) = \lim_{\Lambda \to \mathbb{Z}^2} \alpha_{\Phi,t,\Lambda}(A), \quad A \in \mathcal{A}$ exists and defines a strongly continuous path of automorphisms $\alpha_{\Phi,t}$.
We denote by $\mathrm{QAut}_\beta(\mathcal{A})$ the set of $\alpha \in \mathrm{QAut}(\mathcal{A})$ generated by β-invariant interactions.

SPT in 2-d Fermionic systems

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$\left\{\delta_{(x,y),j}(x,y) \in \mathbb{Z}^2,\ j = 1, \ldots, d\right\}$: standard basis of $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^d$.
$\mathfrak{h}_{(x,y),k} := \mathbb{C} - span\{\delta_{(x,y),2k-1}, \delta_{(x,y),2k}\}$, $(x,y) \in \mathbb{Z}^2$, $k = 1, \ldots \frac{d}{2}$.
$p_{(x,y),k}$: orthogonal projection on $\mathfrak{h}_{(x,y),k}$ onto $\mathbb{C} - \left(\delta_{(x,y),2k-1} + i\delta_{(x,y),2k}\right)$.
$p := \bigoplus_{(x,y),k} p_{(x,y),k}$: a basis projection on $(l^2(\mathbb{Z}^2) \otimes \mathbb{C}^d, \mathfrak{C})$
Set $\omega^{(0)} := \omega_p$ be the Fock state given by p.

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Set $\omega^{(0)} := \omega_p$ be the Fock state given by p.

$$\mathrm{SPT} := \left\{\omega^{(0)}\circ\alpha \mid \alpha\in\mathrm{QAut}(\mathcal{A}),\quad \omega^{(0)}\circ\alpha\circ\beta_g=\omega^{(0)}\circ\alpha.\right\}.$$

We would like to derive some index $h(\omega)$ for each $\omega\in\mathrm{SPT}$ such that

if $\omega_2=\omega_1\circ\alpha$ with $\alpha\in\mathrm{QAut}_\beta(\mathcal{A})$,
then $h(\omega_1)=h(\omega_2)$.

An invariant of 2d Fermi SPT

For $A := \mathbb{Z}_2, \mathrm{U}(1)$, we associate $A \oplus A$ the point-wise multiplication , i.e., for $x = (x_+, x_-), y = (y_+, y_-) \in A \oplus A$, we set $x \cdot y := (x_+ y_+, x_- y_-)$.
Let $a \in H^1(G, \mathbb{Z}_2)$. We define a G-action on $A \oplus A$ by

$$G \times (A \oplus A) \ni (g, x) \mapsto x^{a(g)} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{a(g)} x \in A \oplus A.$$

For $x \in C^1(G, A \oplus A)$, $y \in C^2(G, A \oplus A)$, $z \in C^3(G, A \oplus A)$ and $a \in H^1(G, \mathbb{Z}_2)$, we set

$$d_a^1 x(g,h) := \frac{\left(x^{a(g)}(h)\right) \cdot x(g)}{x(gh)},$$

$$d_a^2 y(g,h,k) := \frac{\left(y^{a(g)}(h,k)\right) \cdot y(g,hk)}{y(gh,k) \cdot y(g,h)},$$

$$d_a^3 z(g,h,k,f) := \frac{\left(\left(z^{a(g)}(h,k,f)\right)\right) \cdot z(g,hk,f) \cdot z(g,h,k)}{z(gh,k,f) \cdot z(g,h,kf)}.$$

For $x = (x_+, x_-) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, we also set $(-1)^x := ((-1)^{x_+}, (-1)^{x_-}) \in \mathrm{U}(1) \oplus \mathrm{U}(1)$.

An invariant of 2d Fermi SPT

By $\widetilde{\mathcal{PD}_0(G)}$, we denote

$$(c, \kappa, a) \in \left(C^3(G, \mathrm{U}(1) \oplus \mathrm{U}(1))\right) \times \left(C^2(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2)\right) \times \left(H^1(G, \mathbb{Z}_2)\right)$$

satisfying

$$d_a^2 \kappa(g, h, k) = 0,$$
$$d_a^3 c(g, h, k, f) = (-1)^{\kappa(g,h) \cdot \left(\kappa^{a(gh)}(k,f)\right)}.$$

For $x = (x_+, x_-) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$, we also set $(-1)^x := ((-1)^{x_+}, (-1)^{x_-}) \in \mathrm{U}(1) \oplus \mathrm{U}(1)$.

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An invariant of 2d Fermi SPT

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We introduce an equivalence relation on $\widetilde{\mathcal{PD}_0(G)}$:

$$(c^{(1)}, \kappa^{(1)}, a^{(1)}) \sim_{\mathcal{PD}_0(G)} (c^{(2)}, \kappa^{(2)}, a^{(2)})$$

if the following hold.
(i)$a^{(1)}(g) = a^{(2)}(g) =: a(g)$ for any $g \in G$, and
(ii) there exist an $m \in C^1(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2)$ and a $\sigma \in C^2(G, \mathrm{U}(1) \oplus \mathrm{U}(1))$ s.t.

$$\kappa^{(2)}(g, h) = d_a^1 m(g, h) + \kappa^{(1)}(g, h),$$

$$c^{(2)}(g, h, k) = (-1)^{\kappa^{(1)}(g,h) \cdot m^{a(gh)}(k) + (m(g)) \cdot \left(\kappa^{(2)}\right)^{a(g)}(h,k)} \, d_a^2 \sigma(g, h, k) c^{(1)}(g, h, k).$$

We denote by $\mathcal{PD}_0(G)$ the equivalence classes.

Theorem (O'21)

There exists a $\mathcal{PD}_0(G)$-valued invariant for 2-d Fermionic SPT.

Split property

Decompose $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^d = \mathfrak{K}_L \oplus \mathfrak{K}_R$ with complex conjugations $\mathfrak{C}_L$, $\mathfrak{C}_R$.

Definition

We say a homogeneous pure state ω on $\mathfrak{A}_{\mathrm{S}DC}(\mathfrak{K}_L \oplus \mathfrak{K}_R, \mathfrak{C}_L \oplus \mathfrak{C}_R)$ satisfies the split property if there are homogeneous states φ_i on $\mathfrak{A}_{\mathrm{S}DC}(\mathfrak{K}_i, \mathfrak{C}_i)$, $i = L, R$ such that ω and $\varphi_L \hat{\otimes} \varphi_R$ are quasi-equivalent.

Remark

$\varphi_L \hat{\otimes} \varphi_R$: *state such that* $(\varphi_L \hat{\otimes} \varphi_R)(a_L a_R) = \varphi_L(a_L)\varphi_R(a_R)$.
quasi-equivalence is physically, macroscopic equivalence.

Our reference state

$$\omega^{(0)} = \omega_p = \omega_{p_L} \hat{\otimes} \omega_{p_R}$$

satisfies the split property.

Split property

Let v_τ be a unitary such that

$$v_\tau \delta_{(x,y),j} := \begin{cases} \delta_{(x,y),j}, & y \neq 0 \\ \delta_{(x,0),j+1}, & y = 0, \quad j = 0, \dots, d-1. \\ \delta_{(x+1,0),1}, & y = 0, \quad j = d. \end{cases}$$

It defines an automorphism τ on $\mathcal{A}$ such that $\tau(B(f)) := B(v_\tau f)$, $f \in \mathfrak{h}$.
$q := v_\tau p v_\tau^*$ defines a basis projection on $(l^2(\mathbb{Z}^2) \otimes \mathbb{C}^d, \mathfrak{C})$.
$\omega^{(1)} := \omega_q$: Fock state given by q also satisfies the split property.
Because $v_\tau p v_\tau^* = q$, $v_\tau q v_\tau^* = p$, $\omega^{(1)} \circ \tau = \omega^{(0)}$ $\omega^{(0)} \circ \tau = \omega^{(1)}$.

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Proposition (O'21)

For any homogeneous pure state ω *on* $\mathfrak{A}_{\mathrm{SDC}}(\mathfrak{K}_L \oplus \mathfrak{K}_R, \mathfrak{C}_L \oplus \mathfrak{C}_R)$ *satisfying the split property, there exists a unique* $a = 0, 1$ *which allows existence of graded automorphisms* $\eta_L \in \mathfrak{A}_{\mathrm{SDC}}(\mathfrak{K}_L, \mathfrak{C}_L)$ *and* $\eta_R \in \mathfrak{A}_{\mathrm{SDC}}(\mathfrak{K}_R, \mathfrak{C}_R)$ *such that*

$$\omega \simeq \omega^{(a)} \circ \left(\eta_L \hat{\otimes} \eta_R\right).$$

Derivation of $\mathcal{PD}_0(G)$-valued invariant

In order to derive the invariant, we consider the restriction β_g^U of our symmetry β_g to the upper half plane.

Let $\omega = \omega^{(0)} \circ \alpha \in \mathrm{SPT}$ with $\alpha \in \mathrm{QAut}(\mathcal{A})$.
From $\omega^{(0)} \circ \alpha \circ \beta_g = \omega^{(0)} \circ \alpha$ and factorization property of $\alpha \in \mathrm{QAut}(\mathcal{A})$, we see that

$\omega^{(0)} \alpha \beta_g^U \alpha^{-1}$ satisfies the split property.

Proposition (O '21)

Let $\omega \in \mathrm{SPT}$. Then there is a unique group homomorpshims $a_\omega : G \to \{0,1\} = \mathbb{Z}_2$ which allows existence of graded $\eta_{g,L}^{\epsilon} \in \mathrm{Aut}\left(\mathcal{A}_{H_L \cap C_\theta}\right)$, $\eta_{g,R}^{\epsilon} \in \mathrm{Aut}\left(\mathcal{A}_{H_R \cap C_\theta}\right)$ such that

$$\omega \circ \beta_g^U \simeq \omega \circ \tau^{a_\omega(g)\epsilon} \left(\eta_{g,L}^{\epsilon} \hat{\otimes} \eta_{g,R}^{\epsilon}\right)$$

for any $0 < \theta < \frac{\pi}{2}$, $\epsilon = \pm 1$ and $g \in G$.

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$$\omega \circ \beta^U_g \simeq \omega \circ \tau^{a_\omega(g)\epsilon}\left(\eta^{\epsilon}_{g,L} \hat{\otimes} \eta^{\epsilon}_{g,R}\right)$$

for any $0 < \theta < \frac{\pi}{2}$, $\epsilon = \pm 1$ *and* $g \in G$.

This $a_\omega : G \to \{0,1\} = \mathbb{Z}_2$ corresponds to a in

$$(c, \kappa, a) \in \left(C^3(G, \mathrm{U}(1) \oplus \mathrm{U}(1))\right) \times \left(C^2(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2)\right) \times \left(H^1(G, \mathbb{Z}_2)\right).$$

Derivation of $\mathcal{PD}_0(G)$-valued invariant

$$\omega \circ \beta_g^U \simeq \omega \circ \tau^{a_\omega(g)\epsilon} \left(\eta_{g,L}^{\epsilon} \hat{\otimes} \eta_{g,R}^{\epsilon}\right)$$

With the factorization property of α, $\alpha = \left(\alpha_L \hat{\otimes} \alpha_R\right) \Upsilon \circ (\text{inner})$, we have $\omega \simeq \left(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R\right) \circ \Upsilon$ which implies

$$\left(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R\right) \circ \Upsilon \circ \beta_g^U \simeq \left(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R\right) \circ \Upsilon \circ \tau^{a_\omega(g)\epsilon} \left(\eta_{g,L}^{\epsilon} \hat{\otimes} \eta_{g,R}^{\epsilon}\right)$$

Setting $\gamma_g^\epsilon := \beta_g^U \left(\eta_{g,L}^{\epsilon} \hat{\otimes} \eta_{g,R}^{\epsilon}\right)^{-1} \tau^{-a_\omega(g)\epsilon}$ for $\epsilon = \pm$, $g \in G$, we have

$$\left(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R\right) \circ \Upsilon \gamma_g^\epsilon \simeq \left(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R\right) \circ \Upsilon.$$

Repeated use of this gives us

$$\left(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R\right) \circ \Upsilon \gamma_g^\epsilon \gamma_h^{(-1)^{a_\omega(g)}\epsilon} {\gamma_{gh}^{\epsilon}}^{-1} \simeq \left(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R\right) \circ \Upsilon.$$

But one can see there is $\zeta_{g,h,\sigma}^\epsilon \in \mathrm{Aut}^{(0)}\left(C_\theta \cap H_\sigma\right)$ s.t.

$$\gamma_g^\epsilon \gamma_h^{(-1)^{a_\omega(g)}\epsilon} {\gamma_{gh}^{\epsilon}}^{-1} = \widehat{\bigotimes}_{\sigma=L,R} \zeta_{g,h,\sigma}^\epsilon$$

Derivation of $\mathcal{PD}_0(G)$-valued invariant

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$$\left(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R\right) \circ \Upsilon \circ \beta_g^U \simeq \left(\omega_L \alpha_L \hat{\otimes} \omega_R \alpha_R\right) \circ \Upsilon \circ \tau^{a_\omega(g)\epsilon} \left(\eta_{g,L}^\epsilon \hat{\otimes} \eta_{g,R}^\epsilon\right)$$

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But one can see there is $\zeta_{g,h,\sigma}^\epsilon \in \mathrm{Aut}^{(0)}\left(C_\theta \cap H_\sigma\right)$ s.t.

$$\gamma_g^\epsilon \gamma_h^{(-1)^{a_\omega(g)}\epsilon} {\gamma_{gh}^\epsilon}^{-1} = \widehat{\bigotimes}_{\sigma=L,R} \zeta_{g,h,\sigma}^\epsilon$$

$$\Rightarrow \quad \omega_R \alpha_R \zeta_{g,h,R}^\epsilon \simeq \omega_R \alpha_R.$$

Derivation of $\mathcal{PD}_0(G)$-valued invariant

$$\gamma_g^{\epsilon} \gamma_h^{(-1)^{a\omega(g)}\epsilon} \gamma_{gh}^{\epsilon\ -1} = \widehat{\bigotimes}_{\sigma=L,R} \zeta_{g,h,\sigma}^{\epsilon}, \quad \omega_R \alpha_R \zeta_{g,h,R}^{\epsilon} \simeq \omega_R \alpha_R.$$

It means that graded automorphism $\zeta_{g,h,R}^{\epsilon}$ is implementable by a unitary $u^{\epsilon}(g,h)$ in the GNS representation π_R of $\omega_R \alpha_R$

$$\mathrm{Ad}\left(u^{\epsilon}(g,h)\right) \pi_R = \pi_R \zeta_{g,h,R}^{\epsilon}.$$

Because $\omega_R \alpha_R$ is homogeneous, there is a self-adjoint unitary Γ_R implementing the grading Θ_R on $\mathcal{A}_R$ i.e.,

$$\mathrm{Ad}\left(\Gamma_R\right) \pi_R = \pi_R \Theta_R.$$

From the fact that $\zeta_{g,h,R}^{\epsilon}$ and Θ_R commute, we can see that $u^{\epsilon}(g,h)$ is graded:

$$\mathrm{Ad}(\Gamma_R)\left(u^{\epsilon}(g,h)\right) = (-1)^{\kappa^{\epsilon}(g,h)} u^{\epsilon}(g,h), \quad \kappa^{\epsilon}(g,h) \in \mathbb{Z}_2$$

This $\kappa^{\epsilon}(g,h) \in \mathbb{Z}_2$ corresponds to κ in

$$(c, \kappa, a) \in \left(C^3(G, \mathrm{U}(1) \oplus \mathrm{U}(1))\right) \times \left(C^2(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2)\right) \times \left(H^1(G, \mathbb{Z}_2)\right).$$

Derivation of $\mathcal{PD}_0(G)$-valued invariant

$$(\omega_L\alpha_L\hat{\otimes}\omega_R\alpha_R)\circ\Upsilon\gamma_g^{\epsilon} \simeq (\omega_L\alpha_L\hat{\otimes}\omega_R\alpha_R)\circ\Upsilon.$$

$$\gamma_g^{\epsilon}\gamma_h^{(-1)^{a_\omega(g)}\epsilon}\gamma_{gh}^{\epsilon\;-1} = \widehat{\bigotimes}_{\sigma=L,R}\zeta_{g,h,\sigma}^{\epsilon}, \quad \omega_R\alpha_R\zeta_{g,h,R}^{\epsilon} \simeq \omega_R\alpha_R.$$

There is a unitary W_g^{ϵ} in the GNS-representation $\pi_L\hat{\otimes}\pi_R$ of $\omega_L\alpha_L\hat{\otimes}\omega_R\alpha_R$ such that

$$\mathrm{Ad}(W_g)\left(\pi_L\hat{\otimes}\pi_R\right) = \left(\pi_L\hat{\otimes}\pi_R\right)\Upsilon\gamma_g^{\epsilon}\Upsilon^{-1}.$$

Using the associativity of automorphisms, it turns out that there is some $c^{\epsilon}(g,h,k)\in\mathrm{U}(1)$ such that

$$W_g^{\epsilon}\left(\mathbb{1}_L\otimes u^{(-1)^{a_\omega(g)}\epsilon}(h,k)\right)W_g^{\epsilon *}\left(\mathbb{1}_L\otimes u^{\epsilon}(g,hk)\right)$$
$$= c^{\epsilon}(g,h,k)\left(\mathbb{1}_L\otimes u^{\epsilon}(g,h)u^{\epsilon}(gh,k)\right)$$

This $c^{\epsilon}(g,h,k)\in\mathrm{U}(1)$ corresponds to c in

$$(c,\kappa,a)\in\left(C^3(G,\mathrm{U}(1)\oplus\mathrm{U}(1))\right)\times\left(C^2(G,\mathbb{Z}_2\oplus\mathbb{Z}_2)\right)\times\left(H^1(G,\mathbb{Z}_2)\right).$$

How can we remove the *doubled structure*?