Recent developments in singularity formation of nonlinear waves.

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Bubbling off dynamics

- Consider an evolution equation of either wave or Schrodinger type

$$-u_{tt} + \triangle u = F(u, \nabla_{t,x} u)$$
$$iu_t + \triangle u = G(u)$$

We are interested in solutions of a 'bubbling type' of essentially the following form

$$u(t,x) = \lambda^{\alpha}(t) Q(\lambda(t)x) + \epsilon(t,x)$$

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- Usually the bulk profile $Q(x)$ is either a stationary or even static solution of the problem.
- Smoothness of solution before blow up : only require smoothness H^s-class in which problem is strongly locally well-posed, i. e. not just C^∞-data.

L^2-Critical NLS

- L^2-critical NLS, for example in one spatial dimension given by

$$iu_t + u_{xx} = -|u|^4 u,$$

admits stationary solution $Q(t,x) = e^{it} \frac{(\frac{3}{2})^{\frac{1}{4}}}{\cosh^{\frac{1}{2}}(\frac{x}{2})}$. Application of suitable pseudo-conformal transformation leads to

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- Application of inherent symmetry leads to a very rigid blow up type (precisely one blow up rate).
- Most 'natural problems' don't admit such inherent algebraic symmetries to infer bubbling off blow up. Nonetheless, the latter is quite ubiquitous.

Models for Bubbling off dynamics

- Key examples which are typically Hamiltonian and also critical :

  Critical Wave Maps : $-u_{tt} + u_{rr} + \frac{1}{r}u_r = \frac{\sin 2u}{2r^2}$

  Critical focussing NLW on $\mathbf{R}^{3+1}$: $-u_{tt} + \triangle u = -u^5$

  Critical Yang-Mills : $-u_{tt} + \triangle u = -\frac{2}{r^2}u(1-u^2)$

  critical Schrodinger Maps : $u_t = u \times \triangle u$

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- Outlier example :

  Hyperbolic Vanishing mean curvature flow :

$$\sum_{\alpha=0}^{n} \partial_\alpha \left(\frac{\partial^\alpha u}{\sqrt{1 + \partial_\alpha u \partial^\alpha u}} \right) = 0, \ n = 8. \tag{1}$$

Bubbling off blow up for WM I

- Specific example : co-rotational critical Wave Maps $\phi : \mathbf{R}^{2+1} \longrightarrow S^2$:

$$-\phi_{tt} + \triangle\phi = \phi(|\phi_t|^2 - |\nabla_x \phi|^2),\ \phi \in S^2 \hookrightarrow \mathbf{R}^3.$$

$$\phi(t,x) = \begin{pmatrix} \cos\theta \sin u \\ \sin\theta \sin u \\ \cos u \end{pmatrix},\ u = u(t,r)\ r = |x|.$$

implies the equation

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- Model admits non-trivial finite energy static solution $Q(r) = 2\arctan r$, corresponding to stereographic projection.

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- Raphael-Rodnianski('09) approach : exhibits open data set within sufficiently smooth class of (co-rotational) data resulting in solutions of the form

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r),\ \lambda(t) = (T-t)^{-1}e^{\sqrt{\log(T-t)}}.$$

The result implies the same blow up rate for an open data set, but the topology is important. The following appears a natural conjecture :

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  Conjecture : Stable finite time blow up solutions of (2) with C^∞-data are of the preceding form.
- One may also conjecture quantized set of blow up rates correspoonding to sufficiently smooth data and unstable blow up.

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- Can be contrasted with the following theorem of Donninger('16) for the ODE-type blow up solutions $u(t,x) = c(T-t)^{-\frac{1}{2}}$ for the energy-critical NLW on $\mathbf{R}^{3+1}$

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- This is the strongest stability statement one can hope for since H^1 is the largest natural space in which the problem is locally well-posed.

The phenomenon of a continuum of blow up rates

- Back to co-rotational critical Wave Maps into S^2, another approach for finite time bubbling off blow up due to K.-Schlag-Tataru('06) : exhibits solutions of the form

  $$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r),\ \lambda(t) = t^{-1-\nu},\ \epsilon \in C^{\nu+\frac{1}{2}-} \cap H^{1+\nu-}.$$

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- Two peculiar features : (1) continuum of blow up rates. (2) The solutions are only of finite regularity, depending on the blow up rate.
- More precisely, the solutions are of class C^∞ in the inside of light cone $|x| < |t|$ centered at singularity, but experience a shock on the light cone $|x| = |t|$.

Remarks on KST('06) construction

- There are two key steps in the construction : (1) construction of an approximate solution u_{approx}. Here the shock on light cone already manifests itself. (2) Completion of approximate solution to an exact one via spectral methods.

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- In the elliptic region, one can make a formal power series ansatz

$$u_{approx} = \sum_{j \geq 0} t^{\nu j} f_j(R),\ R = \lambda(t) r.$$

- In the wave region introduce $a = \frac{r}{t}$, and write

$$u_{approx} = \sum_{j \geq 0} t^{\nu j} g_j(R, a)$$

where the g_j admit suitable Puiseux type expansion in a reflecting the shock across the light cone.

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- To build approximate solution, one needs to distinguish between elliptic region $r \ll t^{\frac{1}{2}}$, the Schrodinger wave region $r \sim t^{\frac{1}{2}}$ and the far region $r \gg t^{\frac{1}{2}}$.
- By analyzing the approximate solution in far region, Perelman can extract the leading radiation part that is left over at the singularity formation.

A further natural candidate which blends wave and Schrodinger

- The critical Zakharov system on $\mathbf{R}^{4+1}$:

$$i\partial_t u + \triangle u = -nu,$$
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- **Conjecture** : Zakharov admits a finite time bubbling off blow up where

$$u(t,x) = e^{i\alpha(t)}\lambda(t)W(\lambda(t)x) + \zeta(t,x),\ \lambda(t) = t^{-\frac{1}{2}-\nu},$$

and $\nu > \nu_* > 0$

How to get a coherent picture of all these dynamics ?

- The issue of stability : the following appears reasonable but non-trivial since due to a nonlinear instability : **Conjecture** : A KST type blow up solution with $\lambda(t) = t^{-1-\nu}$ and ν large is unstable, but stable along a manifold of finite co-dimension in a sufficiently smooth class of perturbations.

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- Recall that these solutions experience shock across light cone of the form $(1-a)^{\frac{1}{2}+\nu}\log(1-a)$, $a = \frac{r}{|t|}$.
- Displacing this shock 'costs a lot', i. e. requires a rough perturbation of the data. Hence natural to consider smooth perturbations which 'cannot displace' the shock.

Stability of KST blow up for critical WM into S^2

- **Theorem**(K.-Miao '19) The KST finite time blow up solutions for critical co-rotational wave maps $u : \mathbf{R}^{2+1} \longrightarrow S^2$ are stable under sufficiently smooth and small *co-rotational* perturbations, provided $\nu > 0$ is sufficiently small. More precisely if $\nu > 0$ is sufficiently small and $u_\nu(t,x)$ a KST blow up solution with $\lambda(t) = t^{-1-\nu}$, constructed on some interval $[t_0, 0)$, and if (ϵ_0, ϵ_1) is sufficiently small in the $H^4 \times H^3$-norm, then the data

$$u_\nu[t_0] + (\epsilon_0, \epsilon_1)$$

lead to a finite time blow up solution of the form

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r)$$

with $\epsilon \in H^{1+\nu-}$. In particular, the perturbed solution blows up in the same space-time location (*rigidity of blow up*).

Comments on result

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- Remarkable feature of co-rotational reduction : no derivatives in nonlinearity :

$$\Box u = \frac{\sin 2u}{2r^2} \text{ versus } \Box u = u(|u_t|^2 - |\nabla_x u|^2).$$

- Applying Duhamel parametrix to source term $\frac{\sin 2u}{2r^2}$ leads to terms of regularity H^{2+}, which gives a key boost in regularity.

Stability of KST blow up under general (non-equivariant) perturbations I

- Up until recently, the stability of either the KST type blow up or the Raphael-Rodnianski blow up for the co-rotational critical Wave Maps into S^2 and under *generic, non-equivariant* perturbations has been completely open. In fact, for the (Ra-Ro) solutions it is conjectured that they are unstable.

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- For the KST blow up u_ν at low ν (where $\lambda(t) = t^{-1-\nu}$ and solutions are at regularity $H^{1+\nu-}$), even if one uses very smooth perturbations (ϵ_0, ϵ_1) of the data, the interactions of u_ν with perturbation in the nonlinearity

$$u(|u_t|^2 - |\nabla u|^2)$$

lead to terms of same regularity as u_ν. Modulations needed, *but only of the kind preserving the locus of the shock.*

Stability of KST blow up under general (non-equivariant) perturbations II

- **Theorem**(K.-Miao-Schlag '20) The KST finite time blow up solutions for critical co-rotational wave maps $u : \mathbf{R}^{2+1} \longrightarrow S^2$ are stable under sufficiently smooth and small *generic* perturbations, provided $\nu > 0$ is sufficiently small. The perturbed solutions are of the form

$$u(t,x) = \mathcal{R}_{h(t)}^{\alpha(t),\beta(t)} \mathcal{L}_{v(t)} \mathcal{S}_{c(t)} \big(Q(\lambda(t) r) + \epsilon(t,x) \big).$$

where $\mathcal{R}_{h(t)}^{\alpha(t),\beta(t)}$ represents a suitable combination of rotations on the target in terms of Euler angles, $\mathcal{L}_{v(t)}$ a suitable Lorentz transform, and $\mathcal{S}_{c(t)}$ a suitable scaling transformation.

Stability of KST blow up under general (non-equivariant) perturbations III

- Key aspects of this work : non-equivariant setting forces one to work in suitable frame for tangent bundle :

$$\begin{pmatrix} \cos\theta\cos U \\ \sin\theta\cos U \\ -\sin U \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

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- If we write $\phi_1 E_1 + \phi_2 E_2$ for the tangential part of perturbation, then. $\phi_1 \pm i\phi_2$ can be decomposed into Fourier series with respect to θ, resulting in

$$\epsilon_{\pm}(n) = \phi_1(n) \pm i\phi_2(n).$$

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- Schrodinger operators

$$H_n^\pm = \partial_{RR} + \frac{1}{R}\partial_R - f_n(R) \pm g_n(R),\ f_n = \frac{n^2+1}{R^2} - \frac{8}{(1+R^2)^2}$$

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$$\phi_n(R;\xi) = h^{\frac{1}{3}}\alpha^{-\frac{1}{2}}q^{-\frac{1}{4}}(\tau)\mathrm{Ai}(h^{-\frac{2}{3}}\tau)(1 + ha_0(-\tau,\alpha,h)).$$

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- For each mode n, one tries to mimic the estimates in the co-rotational case (as in K.-Miao).
- The symmetries of the problem lead to certain algebraic instabilities which manifest in the Fourier modes $n = 0, \pm 1$. This is where the modulations are being used.

Outlook : classification in terms of radiation at blow up time ?

- Recent(2019) work by Jendrej-Lawrie-Rodriguez : write co-rotational WM blow up solution as

$$u(t,r) = Q(\lambda(t)r) + u_*(r) + g(t)$$

where $\lim_{t\to 0} g(t) = 0$. Then if

$$u_*(r) = qr^\nu + o(r^\nu),\ \nu > \frac{9}{2},$$

then $\lambda(t) \sim \frac{|\log t|}{t^{\nu+1}}$.

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then $\lambda(t) \sim \frac{|\log t|}{t^{\nu+1}}$.

- KST solutions have similar radiation part (but also with $\nu > 0$ very small). Probably similar classification ?

Outlook : classification in terms of radiation at blow up time ?

- Recent(2019) work by Jendrej-Lawrie-Rodriguez : write co-rotational WM blow up solution as

$$u(t,r) = Q(\lambda(t)r) + u_*(r) + g(t)$$

where $\lim_{t\to 0} g(t) = 0$. Then if

$$u_*(r) = qr^\nu + o(r^\nu),\ \nu > \frac{9}{2},$$

then $\lambda(t) \sim \frac{|\log t|}{t^{\nu+1}}$.

- KST solutions have similar radiation part (but also with $\nu > 0$ very small). Probably similar classification ?
- How about more general radiation part asymptotics near $r = 0$. More exotic blow up rates ?

Outlook : multibubble solutions ; how much freedom ?

- Precise characterization of two bubble solutions for equivariant wave maps and under a minimal energy condition (threshold blow up) by Jendrej-Lawriew ('20) for $k \geq 2$ and in the co-rotational case $k = 1$ by Rodriguez('18). Rigid blow up rates.

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- In these scenarios only one bubble collapses while the other one converges to a 'limiting bubble'.
- Can there be multi-bubble solutions where all bubbles collapse in finite or infinite time ? This will require more than the threshold energy. Is there a link between the topology one is working with and the possible collapsing rates ?