Moving Frames and their Modern Applications

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The hybrid workshop on moving frames and their applications attracted 56 participants, 8 of which participated in-person at the Banff International Research Station. A total of 25 forty-five minute talks were presented by a wide variety of speakers, from well established experts in the field to more junior researchers. The presentations covered a wide range of topics covering foundational aspects of moving frames, invariants, relative invariants, and their applications to difference equations, differential equations, iterated integrals, numerical schemes, geometry, pedagogy, and more.

1 Overview of the Field

According to Akivis and Rosenfeld, [1], the classical method of moving frames can be traced back to Martin Bartels (1769–1836), a teacher of both Gauss and Lobachevsky, who introduced the concept of moving trihedrons for curves. In the subsequent years, this idea was further developed by Frenet and Serret, and then extended to surfaces by Cotton and Darboux. The classical method of moving frames reached its apotheosis in the first half of the twentieth century when Élie Cartan used his new theory of exterior differential systems to develop a powerful and general method for studying the geometric properties of submanifolds and their invariants under Euclidean, equi-affine, projective, and many other classical groups of transformations, [9].

In 1999, roughly 65 years after the publication of Cartan's pioneering paper [9], Fels and Olver made an important contribution to the field by introducing a novel formulation of moving frames that is applicable to arbitrary finite-dimensional continuous groups of transformations, [10], and to most infinite-dimensional Lie pseudo-group actions, [32]. In the new framework, moving frames are no longer constrained by frame bundles or connections. Instead, given a Lie group G acting on a manifold M, an *equivariant moving frame* is defined as a G-equivariant map $\rho\colon M \to G$ from the manifold M back into the group G (For Lie pseudo-groups, a similar definition holds but it is more technical to state due to the fact that, to date, there is no geometric object that adequately represents an abstract Lie pseudo-group.). The moving frame ρ subsequently yields an explicit invariantization process that maps functions, differential forms, differential operators, differential equations, variational problems, numerical schemes, etc., to their invariant counterparts, which are the keys to solving a wide range of problems in mathematics and natural sciences. By combining the *equivariant moving frame method* with the theory of *variational bicomplexes*, [20], one of the most important new results in the field of moving frames is the derivation of the *recurrence formulas* that enable one to completely determine the structure of the algebra of differential invariants, invariant differential forms, invariant conservation laws, etc., using only linear differential algebra, and, more crucially, without requiring any explicit formulas for either the invariants quantities or the moving frame itself. This feature has led to the introduction of *symbolic* *invariant calculus*, [22], and the development of several new algorithms that can be implemented in symbolic computer packages such as MATHEMATICA, MAPLE, or SAGE.

2 Recent Developments

In the last 22 years, the new equivariant formulation of moving frames has led to a wide variety of novel and unexpected applications in pure and applied mathematics. For example, new algorithms for solving the basic symmetry and equivalence problem of polynomials that form the foundation of classical invariant theory were developed in [4]. The versatility of the equivariant moving frame method has also been used to classify joint invariants and joint differential invariants, establishing a geometric counterpart of what Weyl, in the algebraic framework, calls the first main theorem for the transformation group, [30]. The recurrence formulas discussed in the previous section have produced numerous new results on minimal generating invariants, even in very classical geometries, [17]. The characterization of submanifolds by their differential invariant signatures using moving frames has led to new applications in object recognition and symmetry detection. In the calculus of variations, the problem of directly constructing the invariant Euler–Lagrange equations from their invariant Lagrangians was completely solved using moving frames and the theory of variational bicomplexes, [20]. The combination of these two theories have also been used to study the evolution of differential invariants under invariant submanifold flows, leading to signature evolution in computer vision and integrable bi-Hamiltonian soliton equations, although the underlying reason for the appearance of integrability remains mysterious, [31].

Equivariant moving frames have also been used to compute symmetry groups of partial differential equations and solve the group classification problem [28]; to implement symmetry reduction of dynamical systems; to the automatic assembly of broken objects, including jigsaw puzzles, eggshells, pottery, and bones, of importance in archaeology, anthropology, and surgery, [13, 16]; to recognize DNA supercoils, [33]; to distinguish malignant from benign breast cancer tumors, [14]; for solving the object-image correspondence problem for curves under projections, [8]; to recover the structure of three-dimensional objects from motion, [3]; to classify projective curves in visual recognition, [15]; to construct integral invariant signatures for object recognition in 2D and 3D images, [11]; to design geometric numerical integrators that preserve symmetries and conservation laws, [6, 19, 26]; to determine invariants and covariants of Killing tensors and orthogonal webs, with applications to general relativity, separation of variables, and Hamiltonian systems, [21]. Equivariant moving frames have also clarified the Noether correspondence between symmetries and conservation laws establishing a fully invariant form of Noether's First and Second Theorems, [12, 18]; provided a general tool for computing Casimir invariants of Lie algebras and classifying Lie subalgebras, with applications in quantum mechanics, [7]; and finally moving frames have been used for computing cohomology classes of the invariant variational bicomplex, generalizing earlier work on the projectable case, [35].

In the context of Lie pseudo-groups actions and infinite-dimensional symmetry groups of partial differential equations, the moving frame method has been applied to climate and turbulence modeling, [5]; to partial differential equations arising in control theory, [38]; the classification of Laplace invariants and the factorization of linear partial differential operators, [34]; the construction of coverings and Bäcklund transformations, [29]; and to the method of group foliation, for finding invariant, partially invariant, and other explicit solutions to partial differential equations, [37]. The moving frame calculus also provides a new and very promising alternative to Cartan's exterior differential system approach for solving a broad range of equivalence problems, [2, 27].

Finally, a recent generalization of the theory to discrete equivariant moving frames, [24, 23], has led to new applications to integrable differential-difference systems and invariant evolutions of projective polygons that generalize the remarkable integrable pentagram maps, [25]; and to the extension of the group foliation method to finite difference equations, [36]. In geometric numerical integration, discrete moving frames are also utilized to construct a wide range of numerical integrators that preserve the symmetries of the differential equation being discretized, thereby producing more accurate numerical integrators, especially in challenging regimes where solutions exhibit sharp variations or singularities, [6, 26].

3 Presentation Highlights

Many of the recent developments highlighted in the previous sections were presented at the workshop. The workshop began with a talk by Peter Hydon, in which he introduced the notion of difference moving frame and illustrated its application to invariant Euler–Lagrange equations and the computation of their conservation laws in an invariant form. Discrete moving frames were also used by Gloria Mari-Beffa to construct invariants of nondegenerate twisted polygons in $\mathbb{RP}^{m-1}$. The existence of Hamiltonian evolution equations for these discrete invariants was then discussed. Sehun Chun, showed that moving frames can be used to devise robust numerical schemes with applications in Meteorology, Cardiology, and Neuroscience.

Other novel applications of moving frames include their use by Michael Ruddy to compute invariants of iterated integrals, which have applications in shape analysis, human activity recognition, and more. Next, (algebraic) moving frames were used by Evelyne Hubert to construct rational invariants in classical invariant theory with applications to parameter reduction in Biology and the computation of orthogonal invariants of ternary quartics with application in neuro-imaging.

Applications of moving frames to the integrability theory of differential equations were also widely discussed. Mark Fels eloquently showed how equations of Lie type naturally occur in the context of Darboux integrable equations. A more technical presentation by Thomas Ivey surveyed elliptic Darboux integrable equations and presented open questions related to these equations. Oleg Morozov showed how Cartan's method of moving frames for infinite-dimensional Lie pseudo-groups can be used to obtain differential coverings of partial differential equations admitting infinite-dimensional symmetry groups. These differential coverings can then be used to obtain inverse scattering transformations, bi-Hamiltonian structures, recursion operators, Bäcklund transformations, nonlocal symmetries and nonlocal conservation laws, Darboux transformations, and more.

Foundational aspects of the moving frame method were also discussed during the meeting. Örn Arnaldsson showed how to combine Cartan's method of moving frames with the equivariant moving frame method to obtain, what he dubbed, the method of involutive moving frames. Robert Milson showed how the Karlhede algorithm, used in General Relativity to construct differential invariants of (pseudo-) Riemannian metrics, is linked to Cartan's equivalence method of coframes and how the two approaches can benefit from each other.

Several talks focused on the applications of moving frames to the theory of invariants and the geometric study of (sub-)manifolds. Eivind Schneider gave a talk on differential invariants of Kundt spacetimes and more specifically found generating sets of invariants for three- and four-dimensional degenerate Kundt spacetimes. Boris Kruglikov spoke on relative differential invariants, proving that the graded algebra of (polynomial scalar) relative invariants is finitely generated by a finite number of differential invariants and a finite number of relative invariant derivative operators. Valentin Lychagin computed metric invariants of spherical harmonics, Illia Hayes spoke on joint invariants of primitive actions, Roman Popovych used moving frames to compute generalized Casimir operators of Lie algebras, Emilio Musso presented results on the holomorphic conformal geometry of isotropic curves in the complex quadric, Roman Smirnov showed how to apply moving frames in the theory of orthogonal separation of variables, and finally Dennis The presented a complete Cartan-theoretic classification of multiply-transitive (2,3,5)-distributions, a problem that originated in a famous and challenging paper of Cartan's.

The most unexpected application of moving frames came in Debra Lewis' talk who analyzed the mathematics proficiency of students in STEM fields at the University of California – Santa Cruz. In the context of her study, moving frames were used to compare the equity and efficacy of various processes.

although moving frames were not as directly involved, Mireille Boutin presented an elegant solution to the equivalence problem of noisy unlabeled point configurations under the action of the special Euclidean group. This solution has important applications in computer vision where configuration points are used, for example, in fingerprint or face recognition. In a lively presentation, Tom Needham introduced the Gromov–Wasserstein distance between metric spaces, who's approximations have applications in shape matching and segmentation problems, graph partitioning, and more. To obtain exactly computable lower bounds for the approximation errors, and to verify the quality of solutions, Tom introduced certain distributional invariants of datasets. Lastly, Werner Seiler presented a new rigorous algebraic definition of singularities for systems of differential equations, Ekaterina Shemyakova proposed a construction to obtain super Plücker embeddings, Artur Sergyeyev introduced a large class of new integrable systems in four independent variables with Lax pairs of a novel kind related to contact geometry, and Linyu Peng extended Noether's two Theorems to differential difference variational problems.

4 Scientific Progress and Meeting Outcomes

With the onset of the Covid pandemic, many scientific meetings have been cancelled in the last two years. The BIRS workshop was one of the first opportunities for the broad community of researchers working with moving frames to come together and share their latest results. The wide range of applications considered at the workshop has demonstrated the vitality of the discipline. Several presentations have shown that recent developments in the method of moving frames have opened the doors to many new applications in applied and pure mathematics. Unanticipated and promising applications in Meteorology, Neuroscience, Anthropology, Machine Learning, Pedagogy, and beyond, have demonstrated the impact and tremendous potential of these ideas throughout mathematics, science, and engineering. The broad range of topics covered in the workshop will serve to inspire the research community to develop these research directions further, and to explore new fields of applications.

The talks generated a good amount of questions and discussion, and the in-person participants continued these wide-ranging discussions at meals and other venues, leading to at least one new research project under development. At times, the online participants continued to discuss problems or questions after the presentation, but this exchange of ideas was less robust than it would have been were the participants all on-site. On the other hand, the hybrid format enabled a larger number of researchers world-wide to attend and benefit from the excellent presentations, although time zone issues often prevented full participation for those on other continents. The eight participants that were at Banff made good use of the many opportunities available to discuss mathematics and applications together and in smaller subgroups. In particular, one participant was inspired by such discussions to make significant new progress on a problem that she had been working on before attending the meeting. Three participants used some of the time together to brainstorm ideas for a future conference on symmetry, invariants, and applications in celebration of Peter Olver's 70th birthday next year. Overall, we expect that the interactions and discussions initiated and explored at this meeting will continue and eventually lead to new results and collaborations.

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