Distributions
of
Multitaper Transfer Function Estimates

Multitaper Spectral Analysis (Online)

BIRS 2022 Workshop

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Part 1:

Background & Motivation

Regression in the Time Domain

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$$y(t) = \beta(t)\, x(t) + \varepsilon(t)$$

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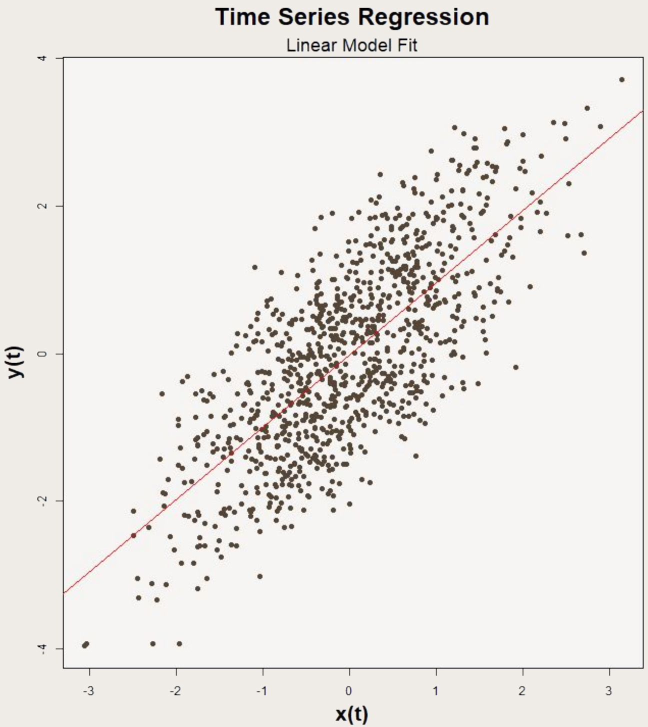
Problem: this model can fail to identify temporal trends!

Example: Time-Correlated Errors



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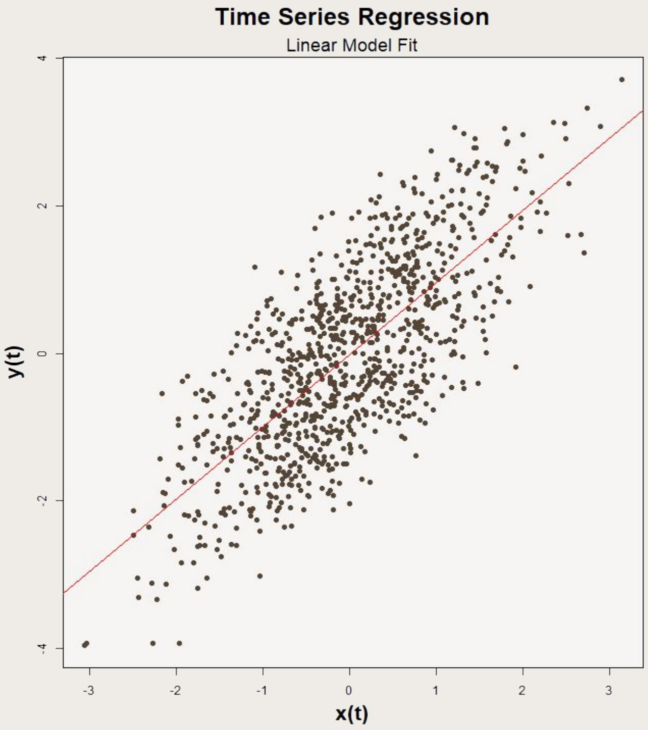

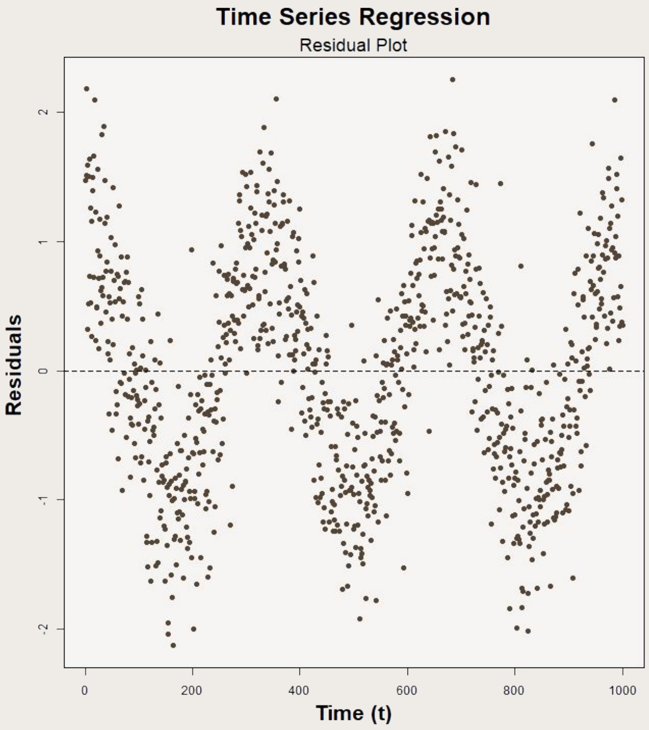

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Part 2:

Distributions of Objects in the Frequency Domain

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$$x(t) = \xi_x(t) + z_x(t) \qquad z_x(t) \sim \mathcal{N}(0, \sigma_x^2)$$

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Assume this noise is strongly stationary.

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$X_k(f)$ is *Complex Gaussian* distributed.

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The stochastic part $Z_k(f)$ has a mean of zero.

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$$\mathrm{E}\left[Z_k(f)\right] = \sum_{k=0}^{N-1} v_k(t)\mathrm{E}\left[z_x(t)\right]e^{-i2\pi f t}$$

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$$\mathrm{E}\left[X_k(f)\right] = \Xi_k(f)$$

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$$\operatorname{Var}\Big[X_k(f)\Big] = \sum_{\substack{s=0\\t=0}}^{N-1} \Big(v_k(s)v_k(t) \; \gamma_x(s-t) \qquad\qquad \Big).$$

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$$\mathrm{Var}\Big[X_k(f)\Big] = \sum_{\substack{s=0\\t=0}}^{N-1} \Big(v_k(s)v_k(t)\ \gamma_x(s-t)\ \cos\left(2\pi f(s-t)\right)\Big).$$

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$$v_k^2(t)$$

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$v_k(s)v_k(t) \rightarrow v_k^2(t)$

$\gamma_x(s-t) \rightarrow \sigma_x^2$

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$$v_k^2(t) \qquad \sigma_x^2 \qquad \cos(0) = 1$$

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There *is* some covariance across *Fourier Frequencies*

$$\mathrm{Cov}\Big[X_k(f), X_k^*(f+\delta)\Big] = \sigma_x^2 \sum_{t=0}^{N-1} e^{i2\pi\delta t} v_k(t)^2.$$

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$$\text{Cov}\left[X_k(f), X_k^*(f+\delta)\right] \neq 0$$

Multitaper Transfer Function Estimates

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$$\hat{H}(f) = \frac{\sum_{j=0}^{K-1} X_j^*(f) Y_j(f)}{\sum_{k=0}^{K-1} |X_k(f)|^2}$$

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Gaussian response time series

→ $\hat{H}(f)$ is complex Gaussian.

Multitaper Transfer Function Estimates: Mean

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Multitaper Transfer Function Estimates: Mean

$$\mathrm{E}\left[\hat{H}(f)\right] = \frac{\sum_{i=0}^{K-1} X_i^*(f)\mathrm{E}\left[Y_i(f)\right]}{\sum_{j=0}^{K-1} |X_j(f)|^2}$$

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The mean is proportional to the eigencoefficient of the pure response signal

Multitaper Transfer Function Estimates: Variance

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Multitaper Transfer Function Estimates: Variance

$$\mathrm{Var}\left[\hat{H}(f)\right] = \frac{\sigma_y^2}{\sum_{j=0}^{K-1} |X_j(f)|^2}$$

Multitaper Transfer Function Estimates: Variance

$$\mathrm{Var}\left[\hat{H}(f)\right] = \frac{\sigma_y^2}{\sum_{j=0}^{K-1} |X_j(f)|^2}$$

The variance is inversely proportional to the spectrum of the predictor

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The MTFE is not frequency stationary.

MTFEs: Moduli

The Rice distribution describes moduli of circularly symmetric CGRVs.

The Modulus of H(f) is Rice distributed with the following parameters

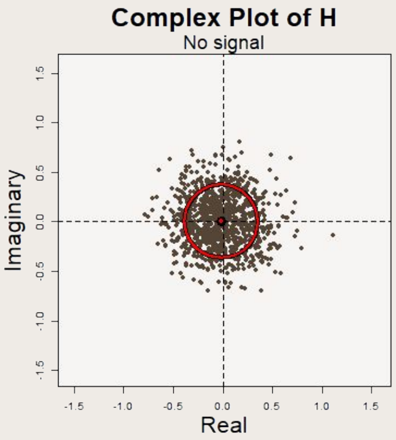
(EQ: 3.51b)

Part 2:

MTFEs at Signal Frequencies

1. Large variance, zero mean

   Indicates no signal is present.



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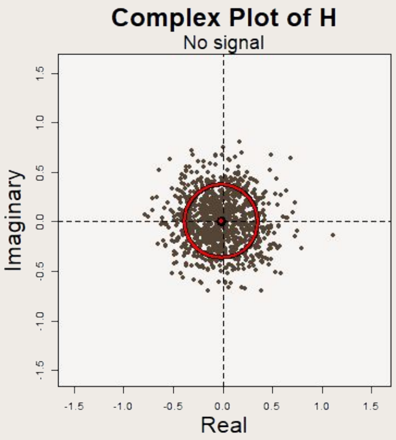

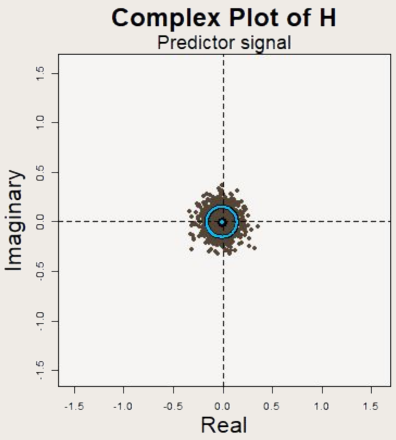

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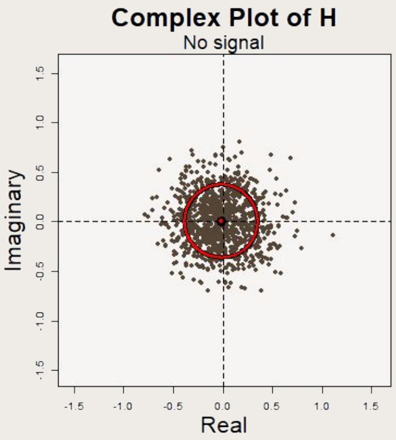
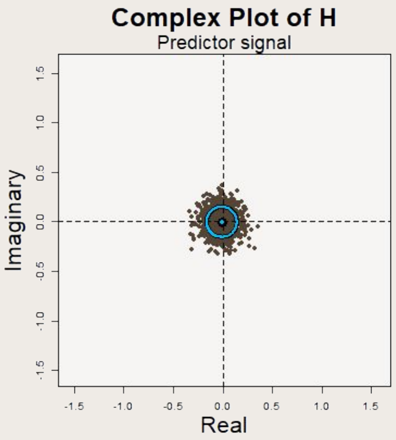
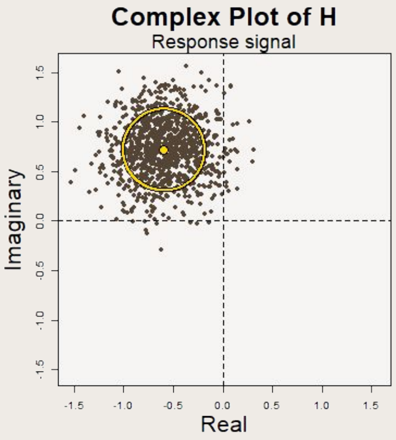
3. Large variance, non-zero mean

   Indicates signal in $y(t)$







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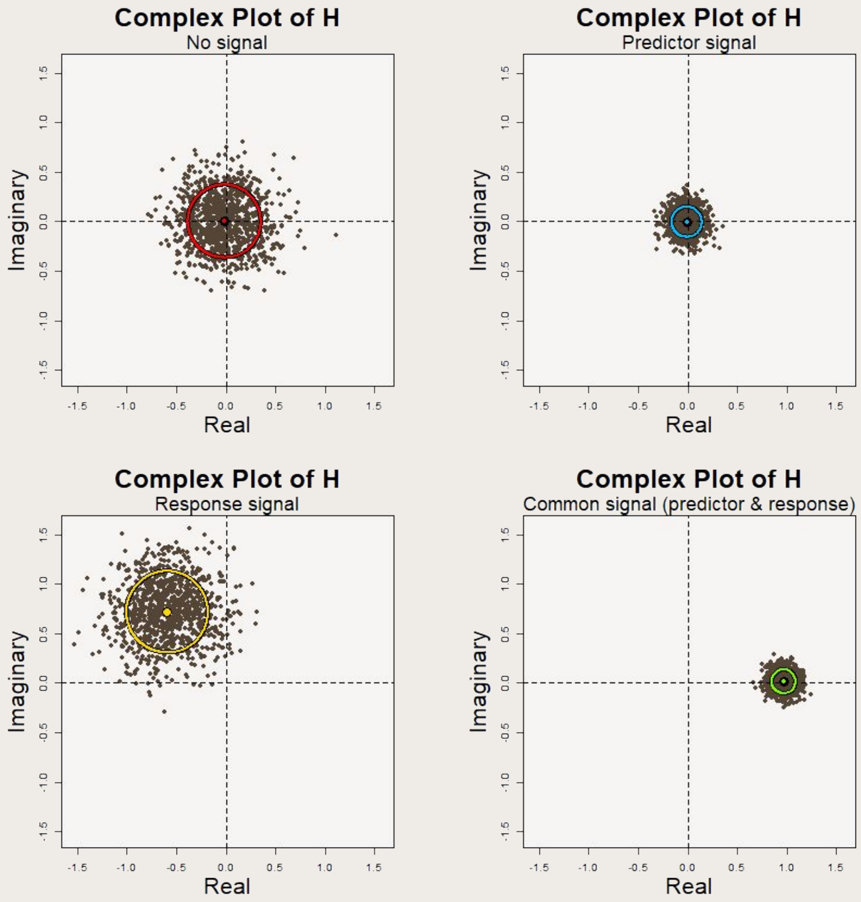
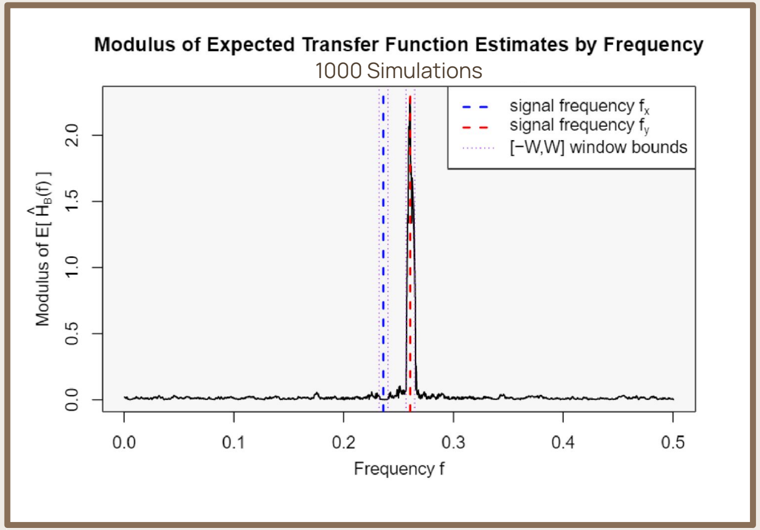
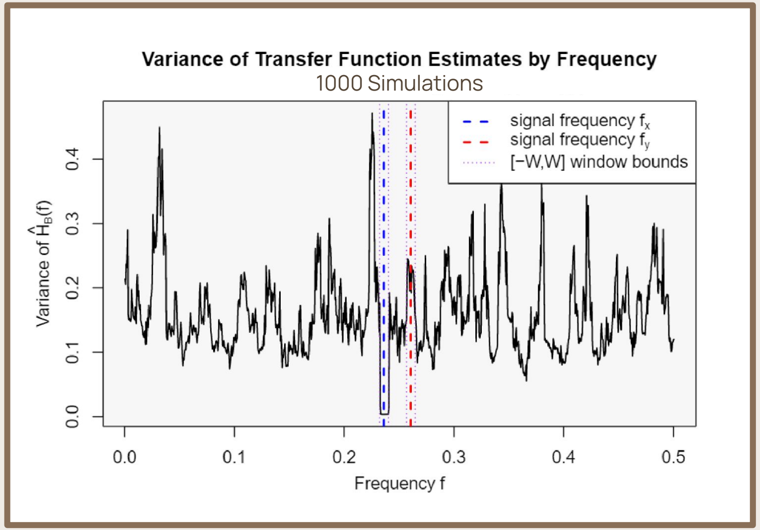
4. Small variance, nonzero mean
   Indicates a *common* signal





Modulus of Expected Transfer Function Estimates by Frequency
1000 Simulations
signal frequency f_x
signal frequency f_y
[−W,W] window bounds
Modulus of $E[\hat{H}_B(f)]$
Frequency f



Variance of Transfer Function Estimates by Frequency
1000 Simulations
signal frequency f_x
signal frequency f_y
[−W,W] window bounds
Variance of $\hat{H}_B(f)$
Frequency f

Test Statistics for Signal Detection

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T_1:

T_2:

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T_1 : $|\mathrm{E}[H(f)]|^2$ detects signals in the response

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This test is less sensitive to “coincidental” coherencies than the MSC.

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This test is more robust to frequency modulation than the Harmonic F-test.

Test Statistics: Definition of T_1

$$T_1(f) = J\left(\tilde{\mu}^2_{\Re}(f) + \tilde{\mu}^2_{\Im}(f)\right) \sim \chi^2_2$$

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J is the number of Fourier Frequencies in a W-Radius Band about f

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$\tilde{H}$ is H, transformed to have uncorrelated entries

Test Statistics: Comparison to MSC

$$MSC_{x,y}^{(m)}(f) = \frac{|\hat{S}_{xy}^{(m)}(f)|^2}{\hat{S}_{xx}^{(m)}(f)\hat{S}_{yy}^{(m)}(f)}.$$

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$$|H(f)|^2 = \frac{|\hat{S}_{xy}^{(m)}(f)|^2}{\left(\hat{S}_{xx}^{(m)}(f)\right)^2}.$$

Test Statistics: “Coherent” Noise

$$Y(f_x^*) = \sqrt{\lambda}\, X(f_x^*) + \sqrt{1-\lambda}\, Z_y^*(f_x^*)$$

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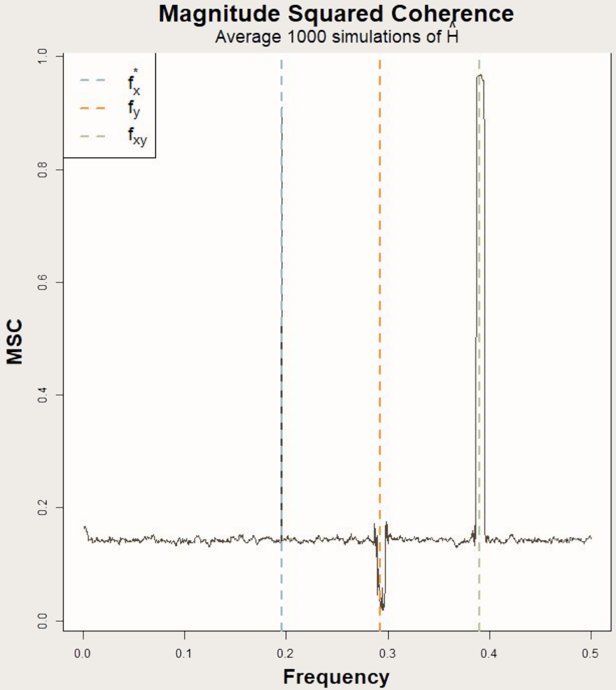

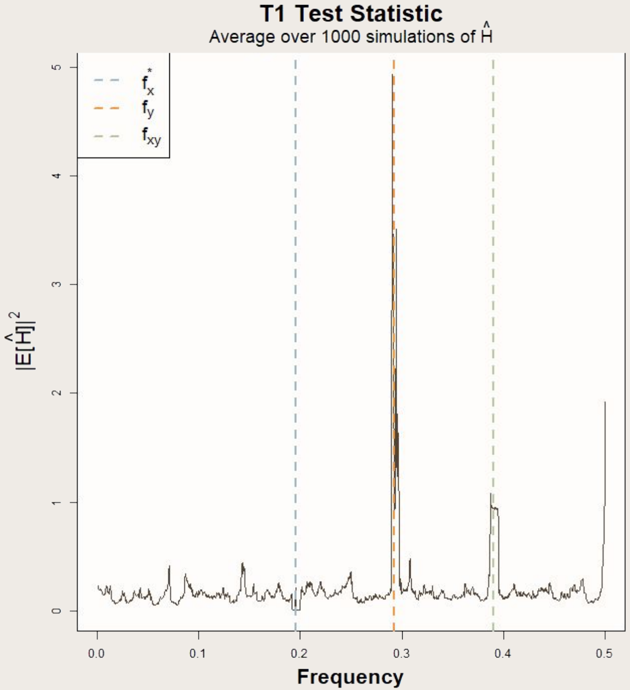

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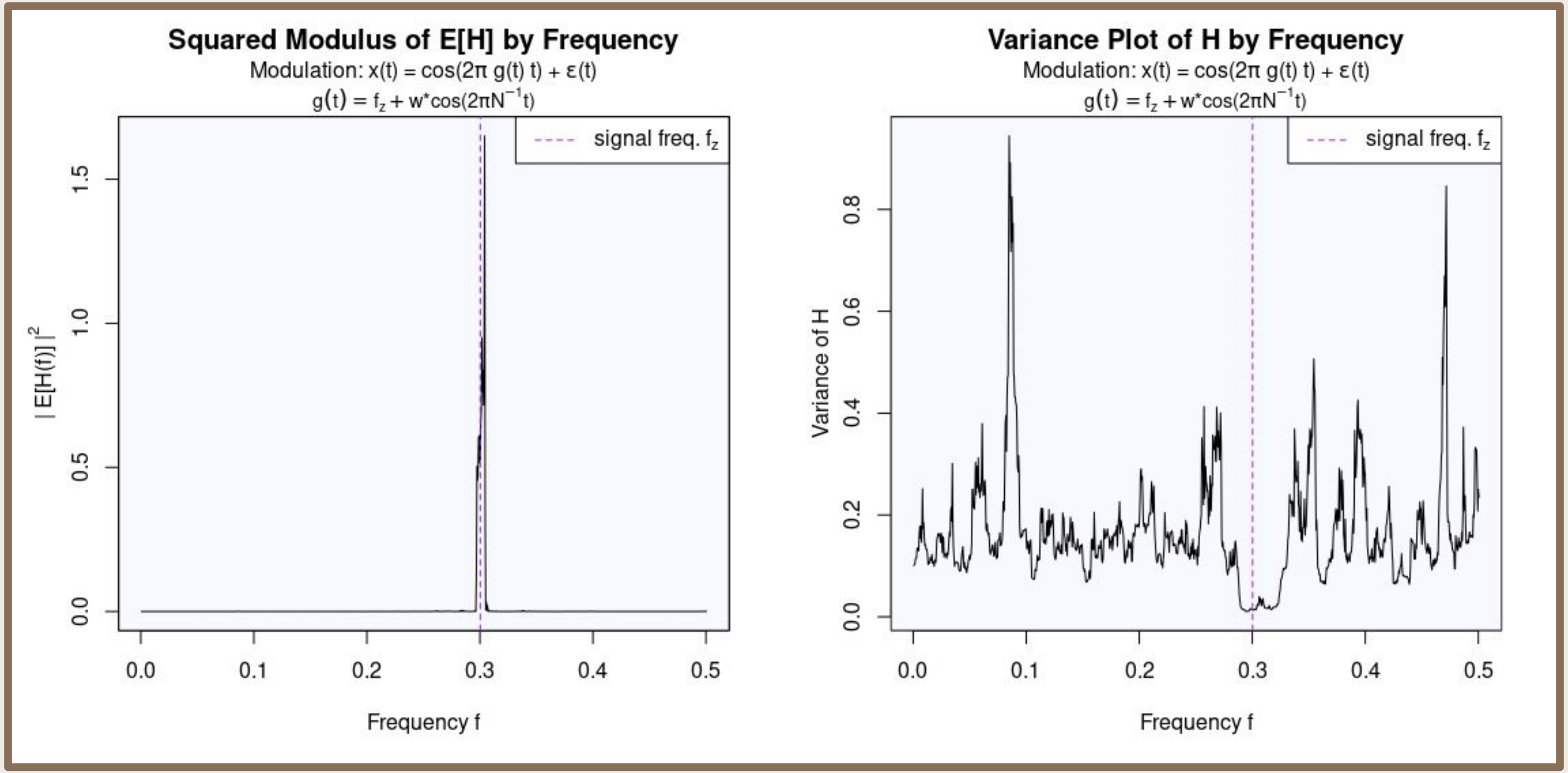
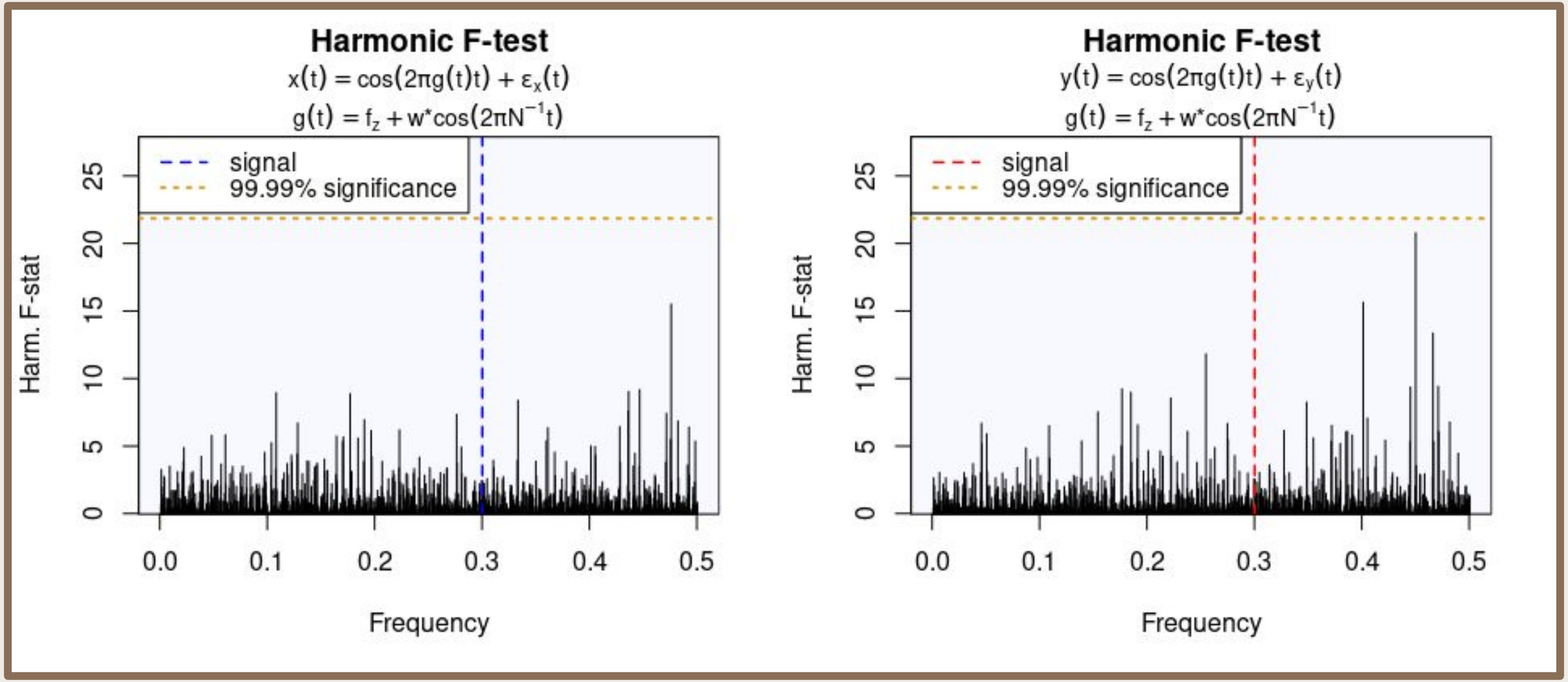
$$y(t) = \cos(2\pi f t) + z_y(t)$$

$$g(t) = f + \cos\left(2\pi A N^{-1} t\right).$$

MTFE: Behaviour under Frequency Modulation



F-Test: Behaviour under Frequency Modulation



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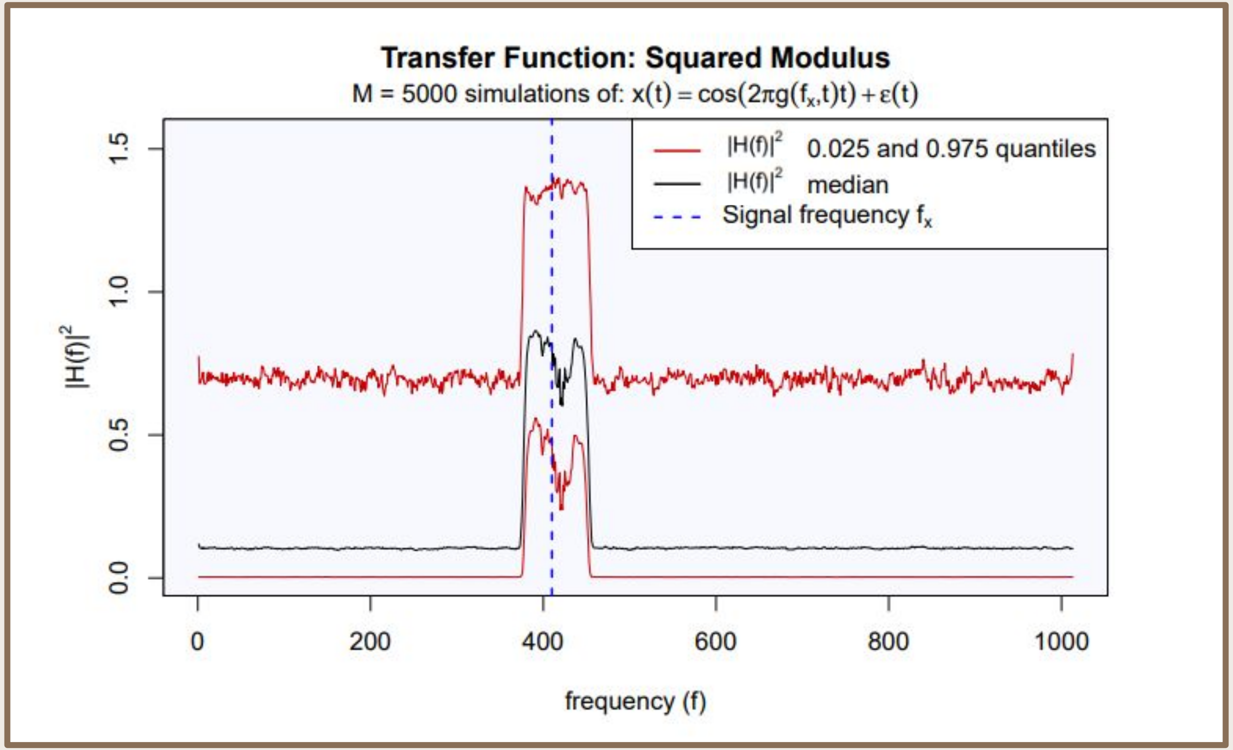
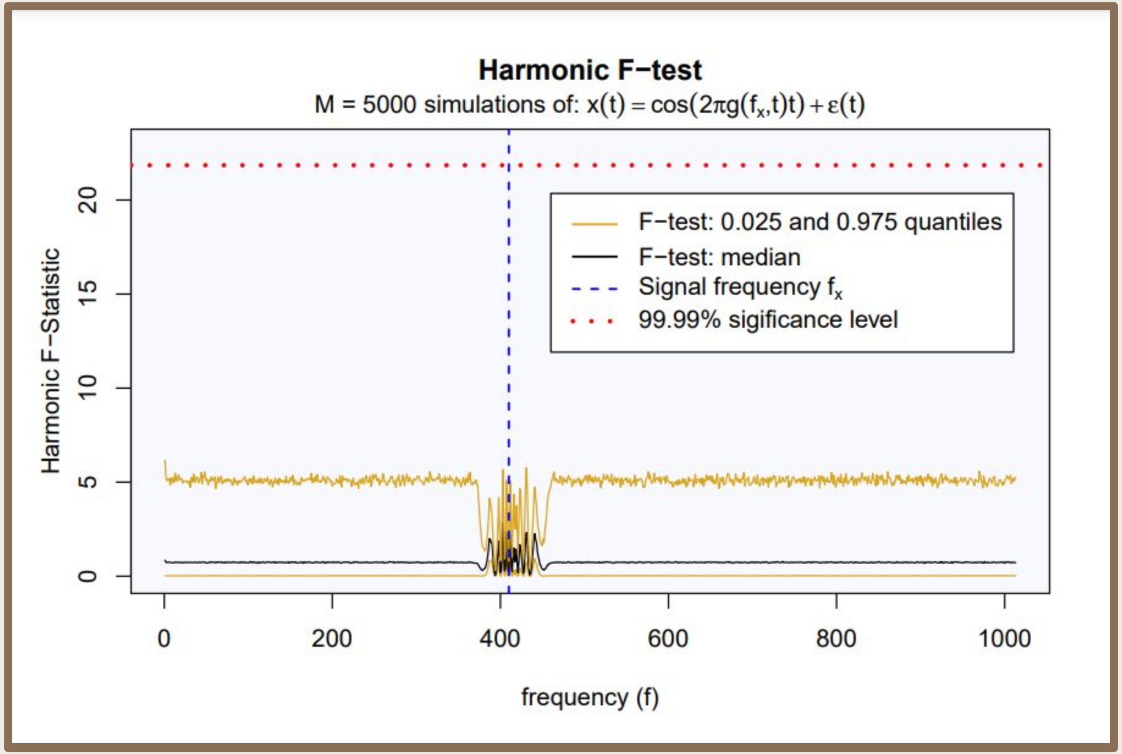


F-Test: Behaviour under Frequency Modulation



Test Statistics for Signal Detection

T_1: $|\mathrm{E}[H(f)]|^2$ detects signals in the response

This test is less sensitive to “coincidental” coherencies than the MSC.

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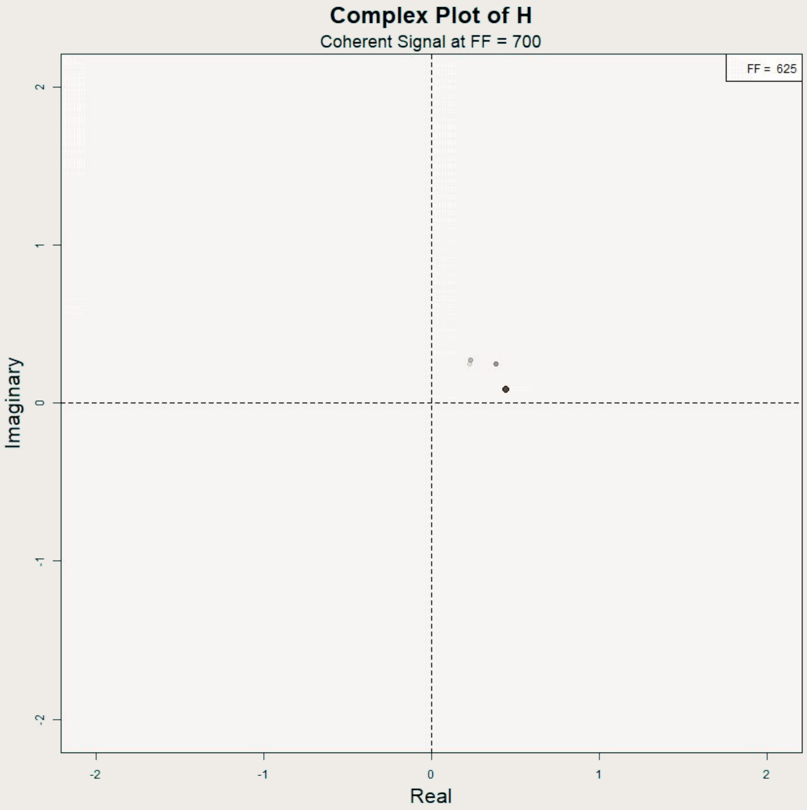
Working with Frequency Non-Stationarity



Part 3:

Phase

Phase Distribution: Complex Gaussian Random Variables

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$$g(\phi|\mu_x, \mu_y, \sigma_x^2, \sigma_y^2) = \frac{\sigma_x \sigma_y \;\exp\left\{-\left(\frac{\mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2}{2\sigma_x^2\sigma_y^2}\right)\right\}\;\left(1 + \sqrt{\pi}\,A(\phi)\exp\left\{A(\phi)^2\right\}\,\mathrm{erfc}\{-A(\phi)\}\right)}{2\pi\left(\sigma_y^2\cos^2(\phi) + \sigma_x^2\sin^2(\phi)\right)}$$

$$\text{where}\quad A(\phi) = \frac{\mu_x\sigma_y^2\cos(\phi) + \mu_y\sigma_x^2\sin(\phi)}{\sigma_x\sigma_y\sqrt{\sigma_y^2\cos^2(\phi) + \sigma_x^2\sin^2(\phi)}}$$

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NOTE: The above distribution assumes x and y are *uncorrelated.*

Phase Distribution: Multitaper Transfer Function Estimates

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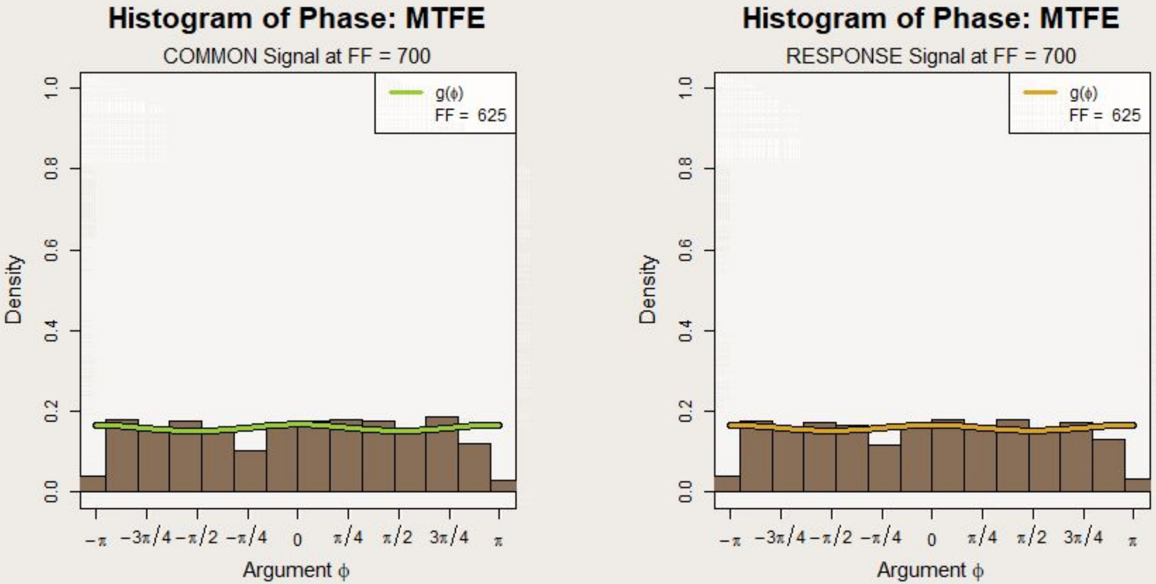

Detecting Signals using the MTFE's Phase

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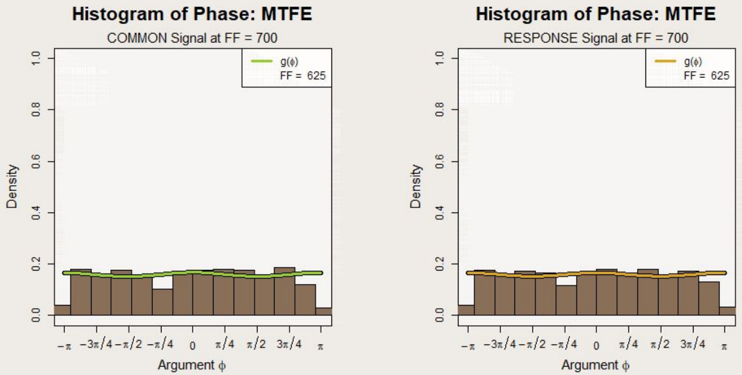
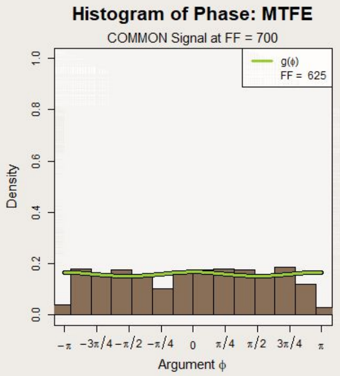
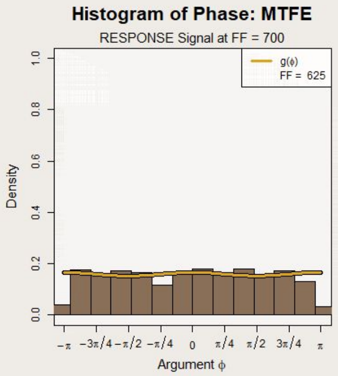


Detecting Signals using the MTFE's Phase

1. Phase distribution will be uniform on $(-\pi, \pi]$ *except at response signal frequencies.*





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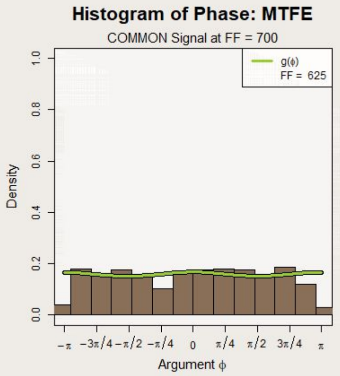

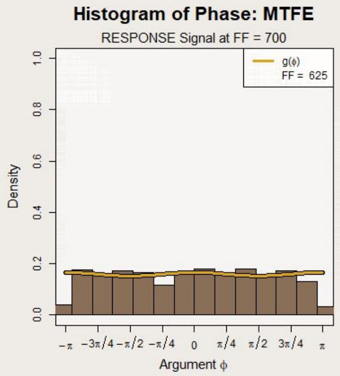

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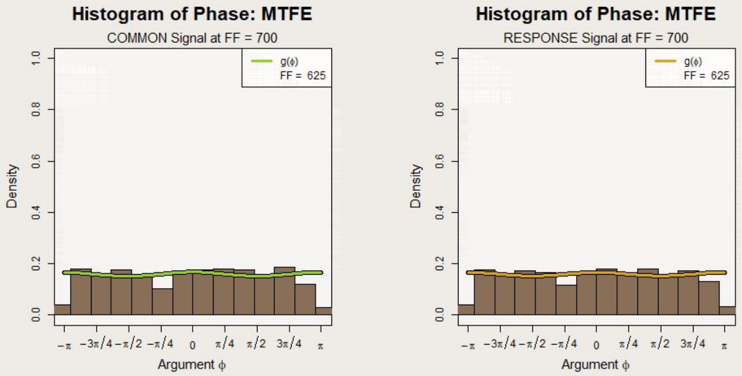

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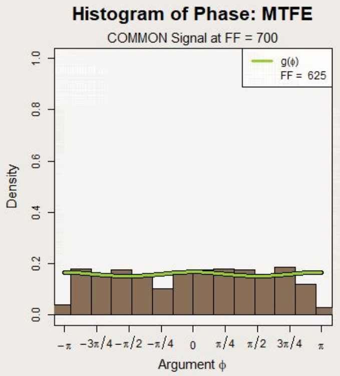
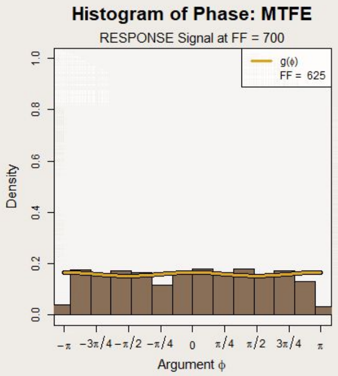
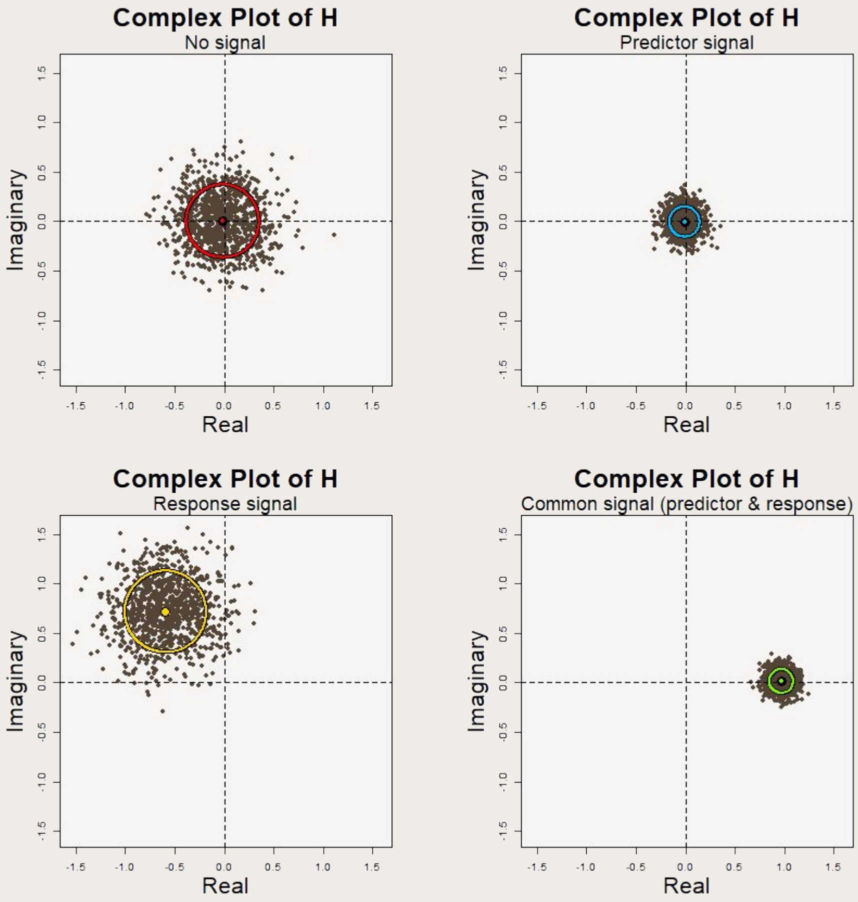
4. If the signal amplitude is large in $y(t)$ compared to $x(t)$, it may be *difficult to distinguish coherent signals from response-only signals.*






Complex Plot of H
No signal
Complex Plot of H
Predictor signal
Complex Plot of H
Response signal
Complex Plot of H
Common signal (predictor & response)
Imaginary
Real
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5

Part 4:

Conclusion & Future Work

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- ★ Extend to other distributions of underlying noise in $x(t)$ and $y(t)$
- ★ What does the phase distribution imply in the *time* domain?

Thanks for Tuning In, Folks ♪

References

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