KE and Egerváry graphs: a stability structure graph decomposition

Mark Kayll
(joint with Jack Edmonds & Craig Larson)

University of Montana

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Banff International Research Station, AB, Canada

4 June 2022

"It's nice to begin a talk with a quote."

Michael Doob 2 October 2004

Kőnig-Egerváry graphs

Definition

G is Kőnig-Egerváry or KE when $\nu = \tau$

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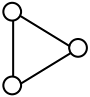
- 

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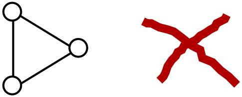
- 

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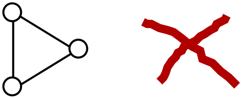
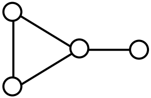
- 
- 

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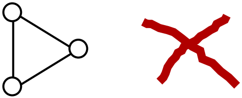
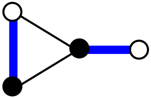
- 
- 

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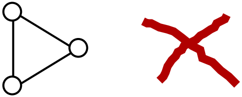
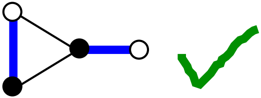
- 
- 

KE literature

Incomplete survey

When	Who	What
1931	Egerváry	bipartite graphs are KE (& more)
1931	Kőnig	″ ″ ″ ″
1979	Deming	characterization (blossom pairs) & algorithm
1979	Sterboul	″ (flowers, posies)
1983	Lovász	″ (ear decompositions)
1986	Lovász, Plummer	″ (neither K_4 nor T_2)
1987	Bourjolly, Pulleyblank	2-bicritical graphs, fractional matchings
2006	Korach, Nguyen, Peis	characterization (extension, forb subgraphs)
2011	Larson	″ (critical independence)
2012	Larson	″ (fractional independence)

Perfect matching polytope

Example

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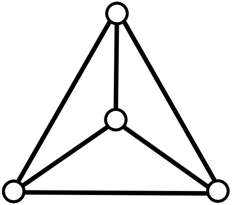

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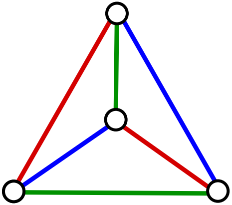

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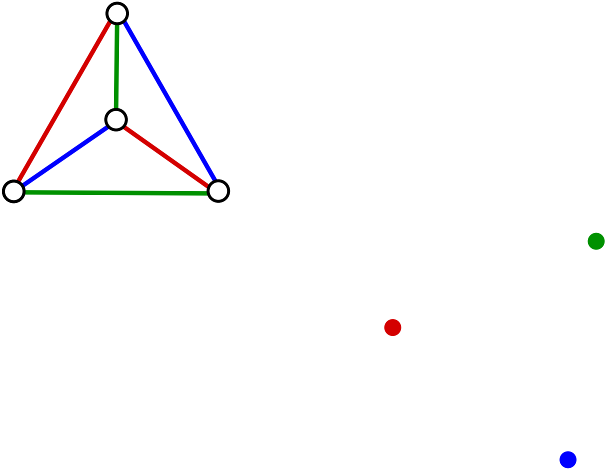

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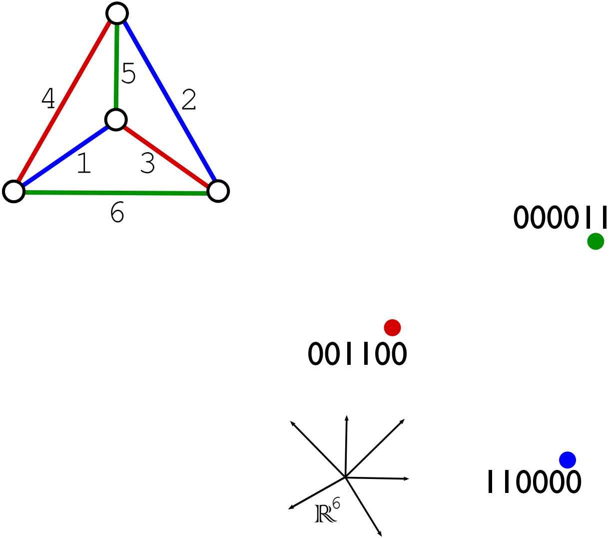

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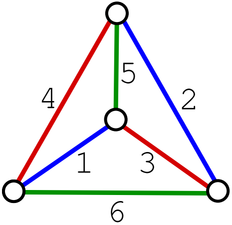

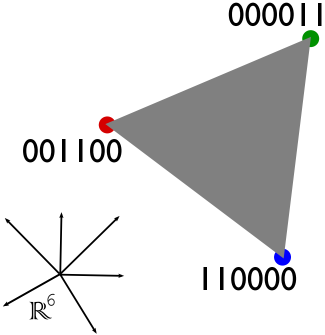

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Official stuff

$$\mathrm{PM}(G) := \mathrm{conv}\left\{ \mathbf{1}_M \;\middle|\; M \text{ is a perfect matching of } G \right\} \subseteq \mathbb{R}^E$$

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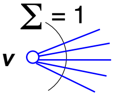

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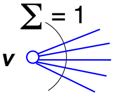

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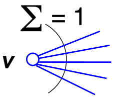

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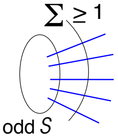

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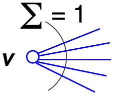

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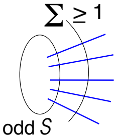

Theorem (Edmonds, 1965)

(i), (ii), (iii) together determine $\mathrm{PM}(G)$

1965....

...good times...

Outline from here

Preamble: Doob joke

Warm-up

- Kőnig-Egerváry graphs
- Perfect matching polytope
- 1965

Egerváry graphs

- Basics
- LP & characterizations
- Connections: bipartite, KE, Egerváry

KE graphs

- Stable sets
- Deming's Algorithm & extensions
- Deming decompositions

More on Egerváry graphs

- Constructions
- A conjecture

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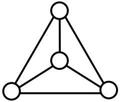
- 

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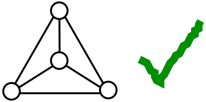
- 

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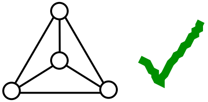
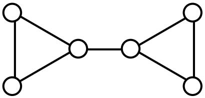
- 
- 

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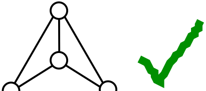
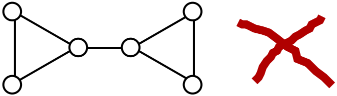
- 
- 

Egerváry literature

Incomplete survey

When	Who	What
1936	Kőnig	bipartite graphs are Egerváry (& more)
1946	Birkhoff	″ ″ ″ ″
1953	von Neumann	″ ″ ″ ″
1953	Hoffman, Wielandt	″ ″ ″ ″
1956	Hammersley, Mauldon	″ ″ ″ ″
1981	Balas	characterize (forbidden config $C_1 \cup C_2 \cup M$)
2004	de Carvalho, Lucchesi, Murty	characterization (solid bricks)
2010	Kayll	example class of Egerváry graphs
2020	de Carvalho, Lin, Kothari, Wang	PM-compactness
2012–	Edmonds, Kayll, Larson	today's talk

Corollary

A matchable G is Egerváry

$$\Longleftrightarrow$$

G admits no spanning subgraph

$$M \cup \bigcup_{k=1}^{2\ell\ (\geq 2)} C_k$$

Characterizations (matchable case)

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Egerváry

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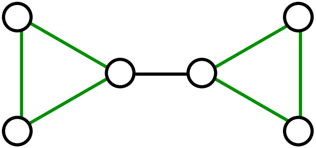

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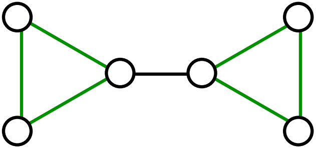

G contains no nice even subdivision of T_2

Proof...

What's needed

(for matchable G)

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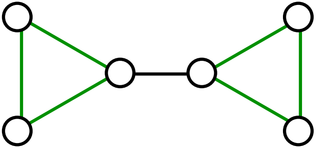

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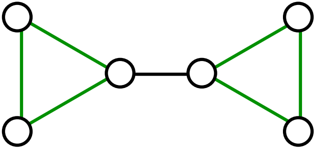

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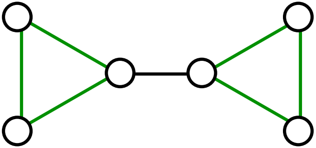

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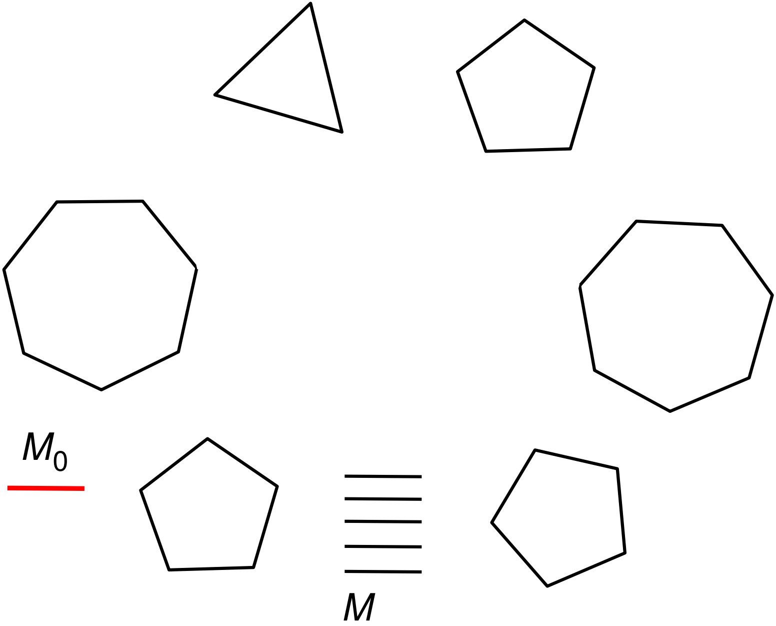

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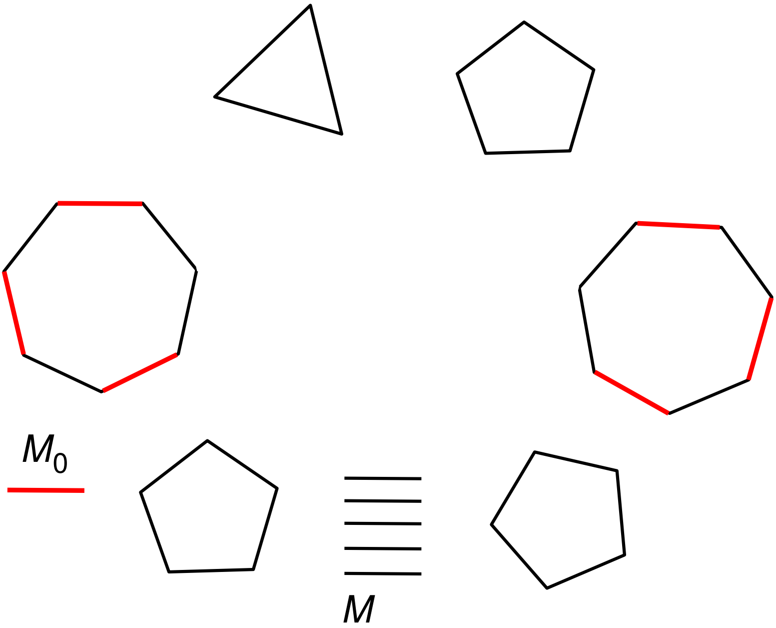

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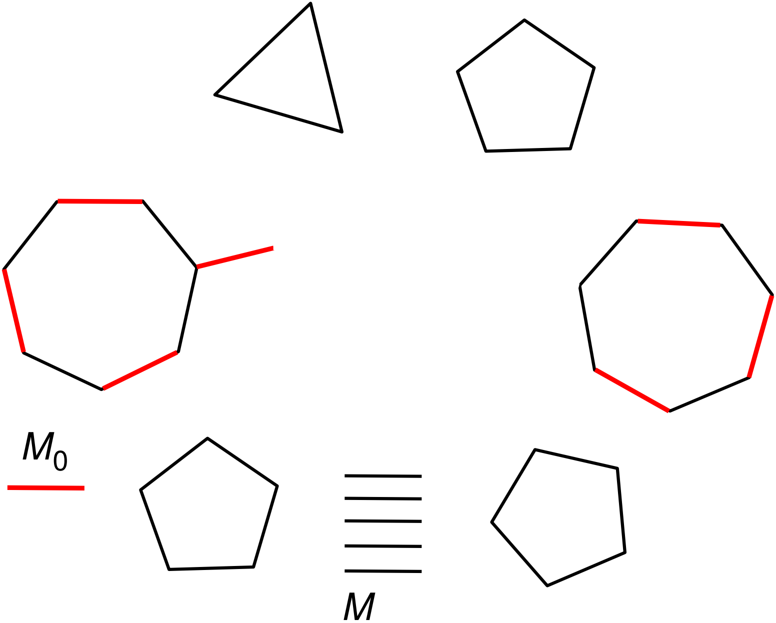

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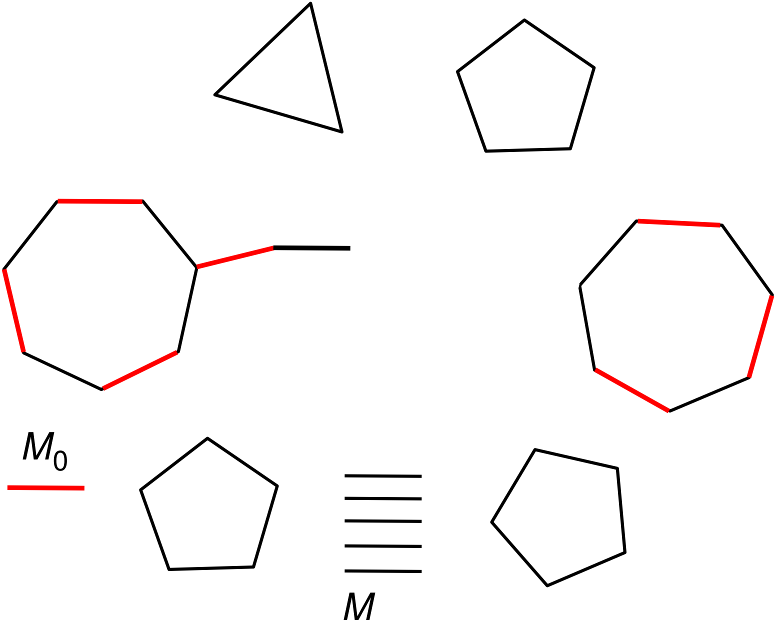

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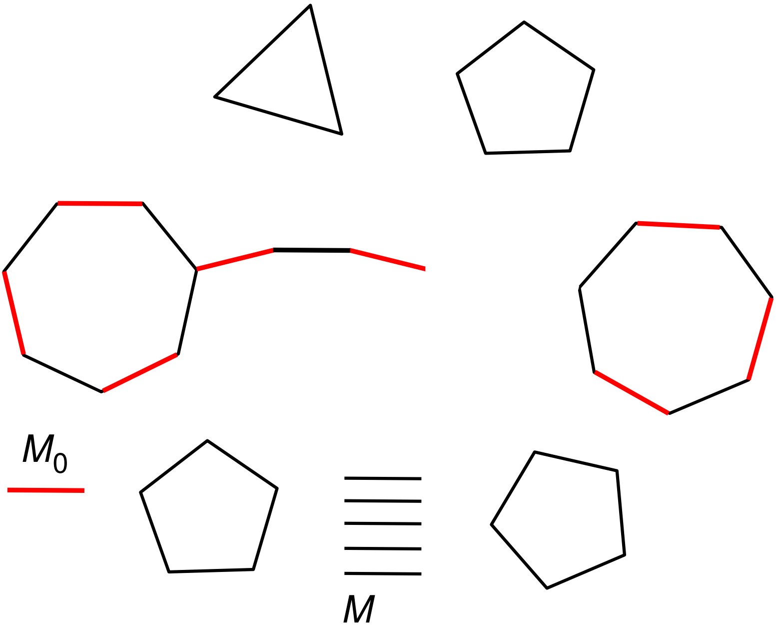

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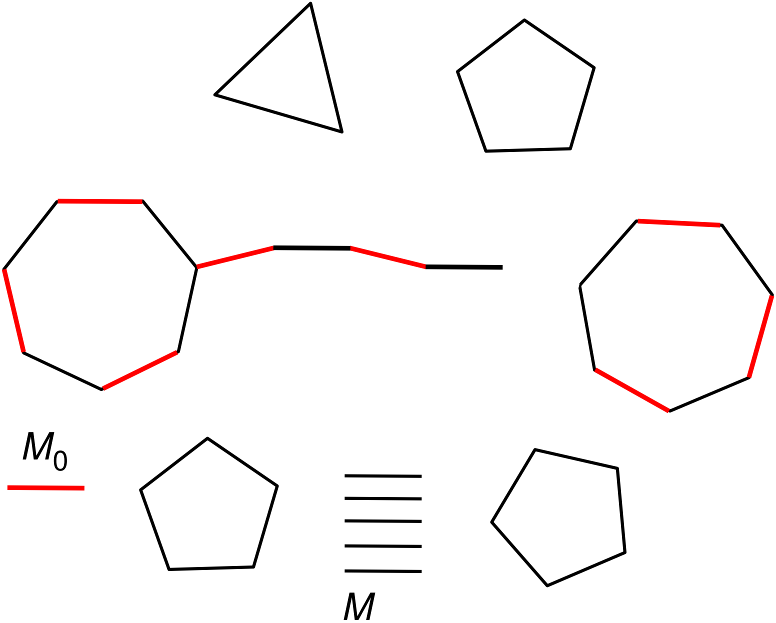

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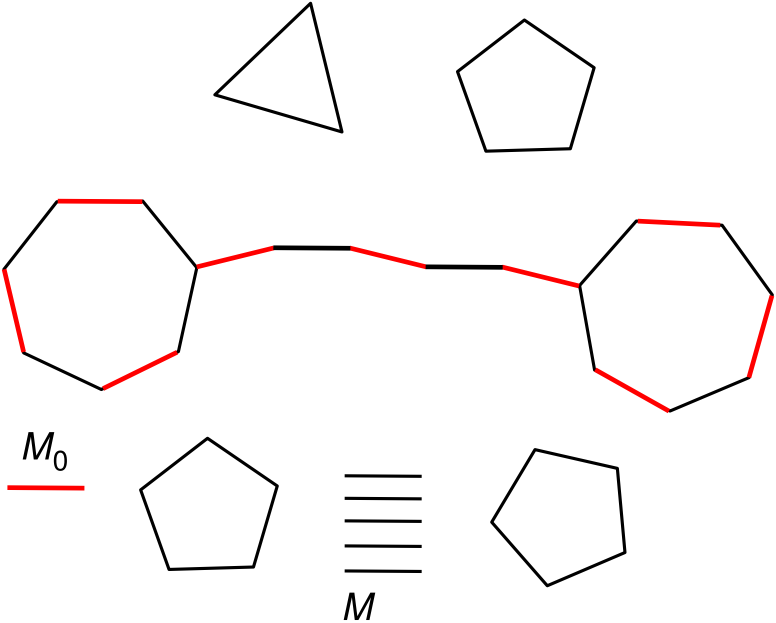

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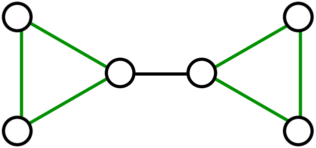

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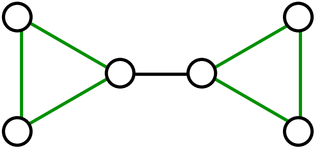

QED

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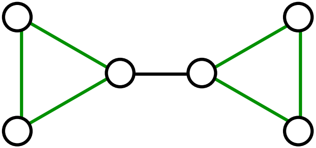
Egerváry



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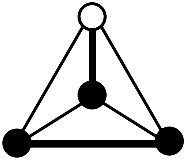

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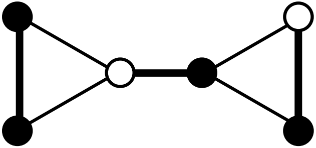

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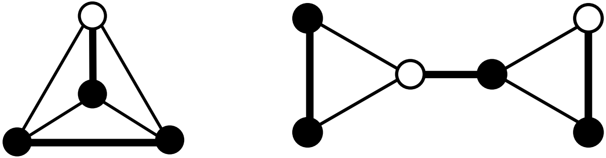

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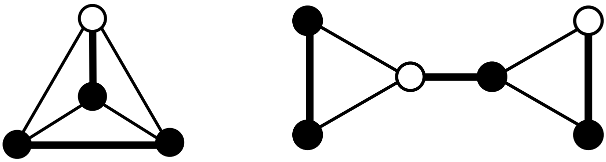

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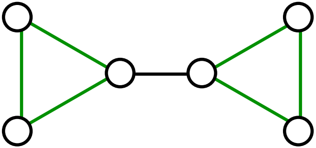

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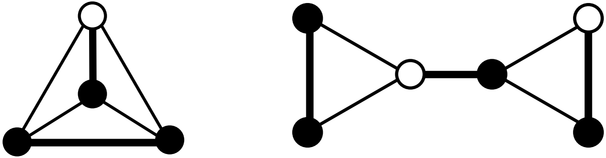

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Two big theorems

Kőnig–Egerváry (1931,1931)

Bipartite graphs are KE

Birkhoff–von Neumann (1946,1953)

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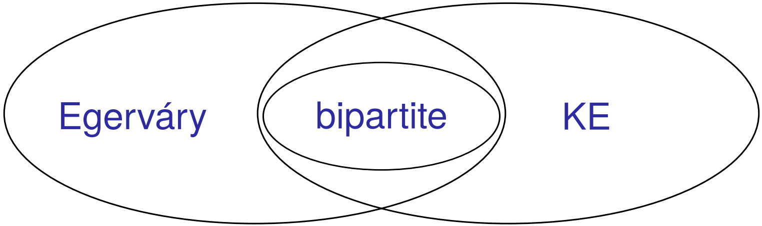


Two big theorems plus a little one

Kőnig–Egerváry (1931,1931)

Bipartite graphs are KE

Birkhoff–von Neumann (1946,1953)

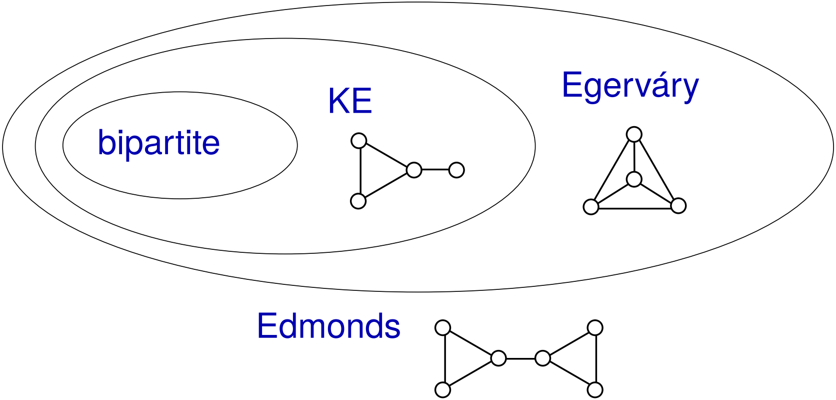
Bipartite graphs are Egerváry

Kayll (2010)

KE graphs are Egerváry

Four graph classes

(actual state of affairs)



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- KE graphs are in NP:
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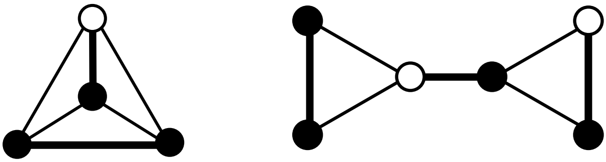
- Are they in co-NP?
- Can we find a maximum stable set efficiently?

Characterizations (matchable case)

KE



G contains no nice even subdivision of K_4 or T_2

KE graphs: towards algorithmics

Again: G is KE

$$\iff$$

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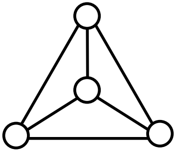

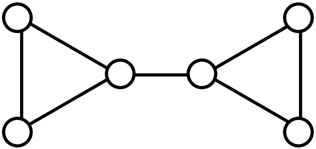

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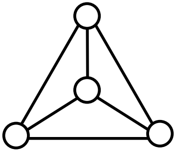

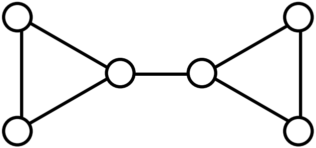

Parameters: $\nu = \frac{n}{2}$

KE graphs: towards algorithmics

Again: G is KE

$$\Longleftrightarrow$$

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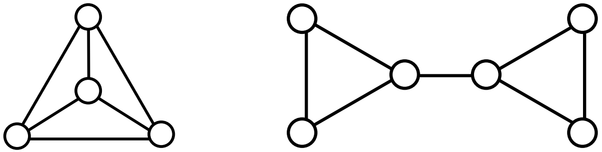

Parameters: $\nu = \frac{n}{2}$

$\alpha = \nu - 1$

KE graphs: towards algorithmics

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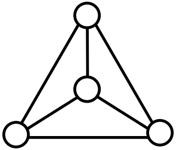

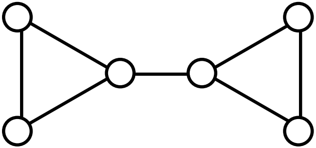

Parameters: $\nu = \frac{n}{2}$

$\alpha = \nu - 1$

(so $\alpha + \nu < n$)

Deming's Algorithm (1979)

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INPUT: matchable G of order n

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OUTPUT: **either** a stable set of $\frac{n}{2}$ nodes (so $\alpha + \nu = n$ ... KE)

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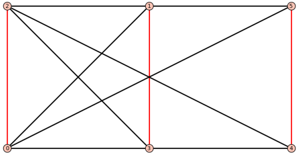
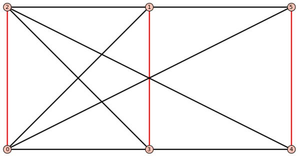
ALSO: efficient computation of ν, α for KE graphs

(including *unmatchable* ones)

Extending Deming's Algorithm

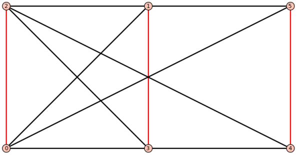


Extending Deming's Algorithm



- G not KE $\Rightarrow G$ has a K_4 or T_2 obstruction H

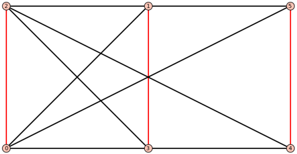
Extending Deming's Algorithm



- G not KE $\Rightarrow G$ has a K_4 or T_2 obstruction H
- S max stable set, M perfect matching, $|S| < |M|$

  $\implies$ some $e \in M$ meets no $v \in S$

Extending Deming's Algorithm

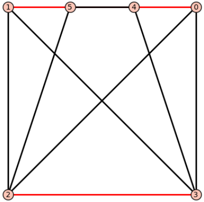


- G not KE $\Rightarrow G$ has a K_4 or T_2 obstruction H
- S max stable set, M perfect matching, $|S| < |M|$

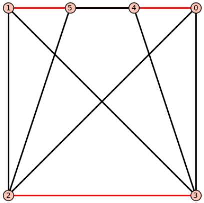
  $\Longrightarrow$ some $e \in M$ meets no $v \in S$
- with $e = xy$ **either** $H - \{x, y\}$ is KE

  or it (still) has an obstruction

Extending Deming's Algorithm

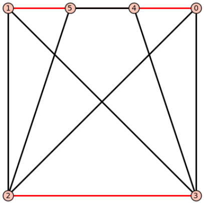


Extending Deming's Algorithm



- So: with M a perfect matching of H & each $xy \in M$ run Deming on $H - \{x, y\}$

Extending Deming's Algorithm

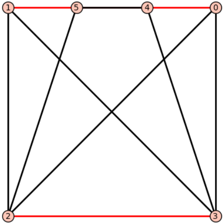


- So: with M a perfect matching of H & each $xy \in M$ run Deming on $H - \{x, y\}$

Definition H is a **Deming subgraph** if each $H - \{x, y\}$ is KE

Deming graphs are almost-KE

H



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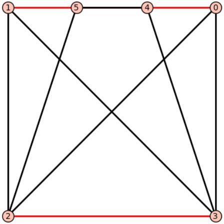

- H contains a spanning even K_4-subdivision

Deming graphs are almost-KE

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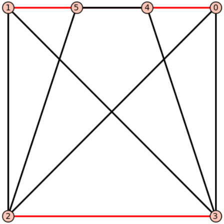

- H contains a spanning even K_4-subdivision
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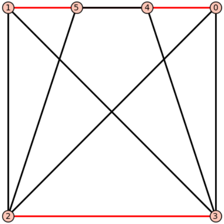
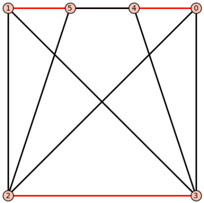


- H contains a spanning even K_4-subdivision
- removing ends of any red edge yields a KE graph
- $\alpha = 2$ and $\nu = 3$

Extending Deming's Algorithm



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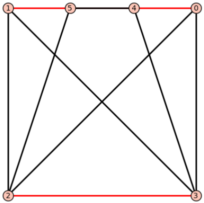


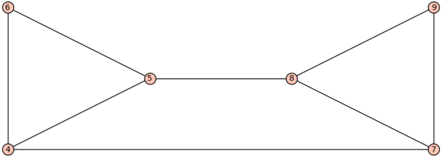
Definition H is a **Deming subgraph** if each $H - \{x, y\}$ is KE

- Extended Deming Algorithm

  Either G is KE or it contains a Deming subgraph H; repeat algorithm on $G - H$

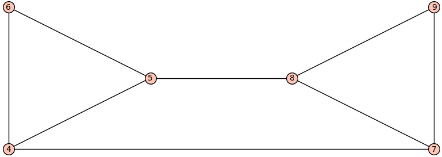
Deming subgraphs

D



Deming subgraphs

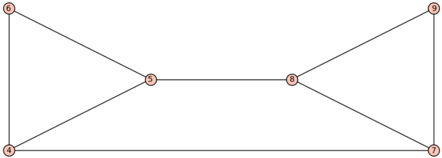
D



- *D* contains a spanning even T_2-subdivision

Deming subgraphs

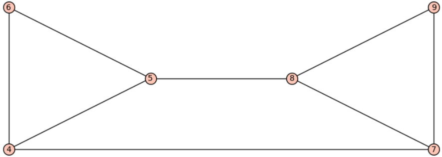
D



- D contains a spanning even T_2-subdivision
- Deming subgraphs are not KE, contain a spanning even subdivision of K_4 or T_2, and have $\alpha = \nu - 1$

Deming subgraphs

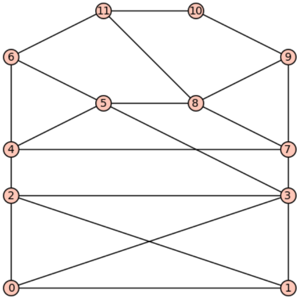
D



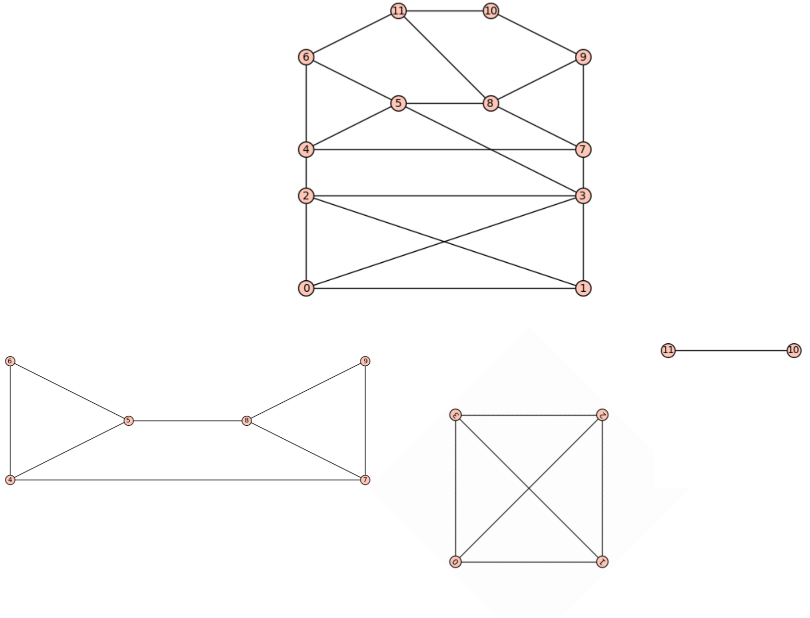
- D contains a spanning even T_2-subdivision
- Deming subgraphs are not KE, contain a spanning even subdivision of K_4 or T_2, and have $\alpha = \nu - 1$

Definition these are **Deming-K_4** or **Deming-T_2** subgraphs

Deming decomposition: example



Deming decomposition: example



Deming decomposition

Deming decomposition

- What?

Deming decomposition

- What? decomposition of a matchable G into Deming subgraphs $\{K_i\}_{i=1}^{\ell}$, $\{T_j\}_{j=1}^{t}$ plus a KE subgraph R

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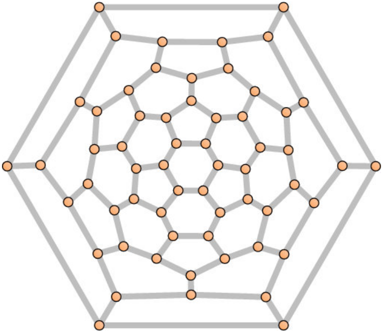
- Stability?

$$\alpha(G) \leq \alpha(R) + \sum_{i=1}^{\ell} \alpha(K_i) + \sum_{j=1}^{t} \alpha(T_j)$$

Buckminster Fullerene C_{60}



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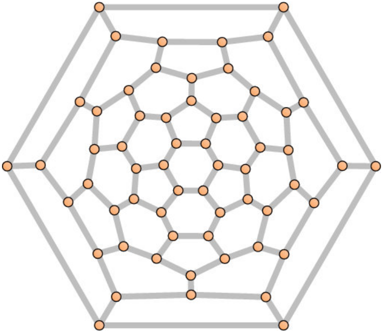
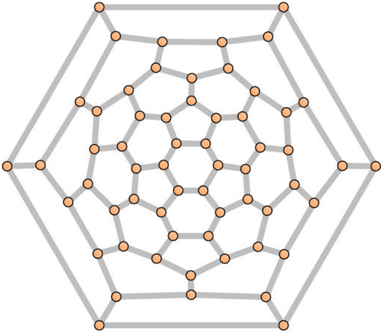


- $n = 60$ and there are 12 pairwise-disjoint pentagonal faces

Buckminster Fullerene C_{60}



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- A Deming decomp: 6 pairs of C_5's joined by a single edge

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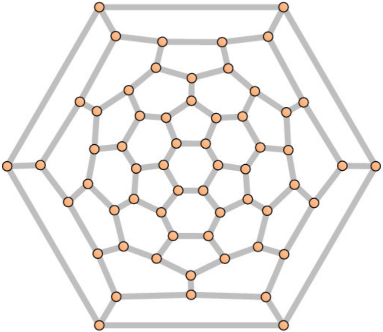
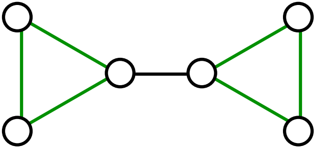


- $n = 60$ and there are 12 pairwise-disjoint pentagonal faces
- A Deming decomp: 6 pairs of C_5's joined by a single edge
- $$\alpha(C_{60}) = 24 = 4 + 4 + 4 + 4 + 4 + 4 = \sum_{j=1}^{6} \alpha(T_j)$$

Characterizations (matchable case)

Egerváry

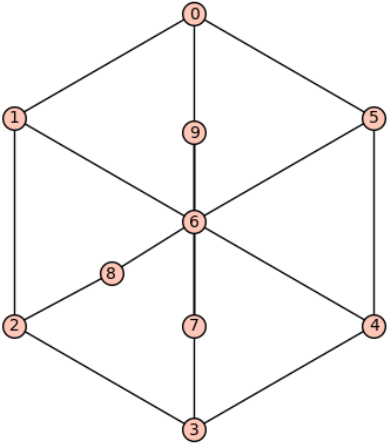


G contains no nice even subdivision of T_2

Egerváry constructions

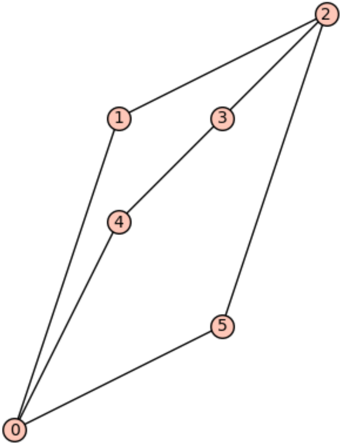
Egerváry constructions

Weak wheels



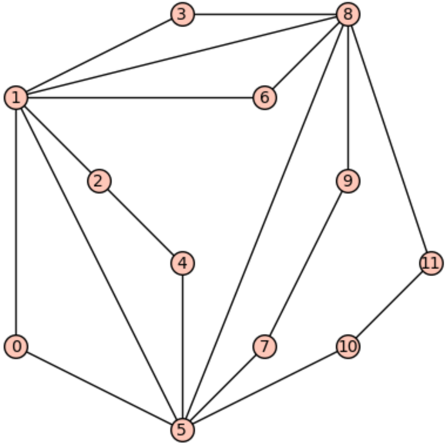
Egerváry constructions

Weak bananas



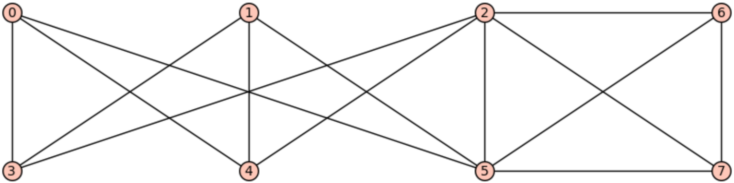
Egerváry constructions

Bracelets



Egerváry constructions

Bipartite extensions



Corollary

Corollary

Weak wheels, weak bananas, bracelets, and bipartite extensions are all Egerváry.

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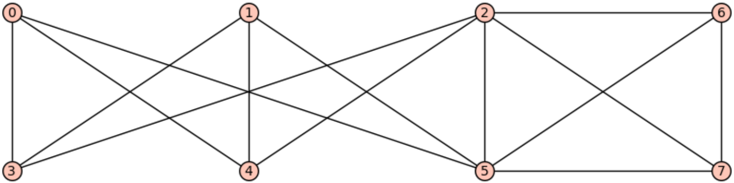
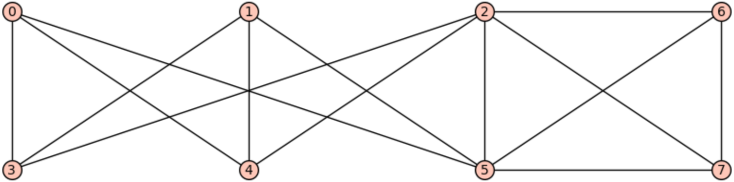
Weak wheels, weak bananas, bracelets, and bipartite extensions are all Egerváry.

Proof: These graphs don't contain disjoint odd cycles. □

Egerváry Graph Conjecture

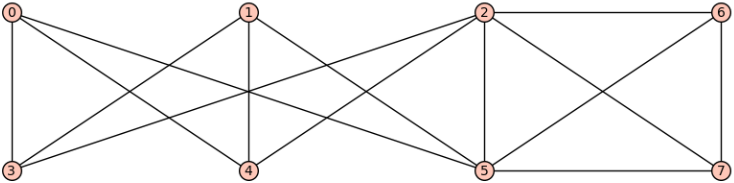


Egerváry Graph Conjecture



$\alpha = 3$; a Deming decomposition consists of the K_4 (with $\alpha = 1$) plus the graph induced on $\{0, 1, 3, 4\}$ (with $\alpha = 2$)

Egerváry Graph Conjecture



$\alpha = 3$; a Deming decomposition consists of the K_4 (with $\alpha = 1$) plus the graph induced on $\{0, 1, 3, 4\}$ (with $\alpha = 2$)

Conjecture

G Egerváry $\implies$ α is additive on its Deming decomposition:

$$\alpha(G) = \alpha(R) + \sum_{i=1}^{\ell} \alpha(K_i) + \sum_{j=1}^{t} \alpha(T_j)$$