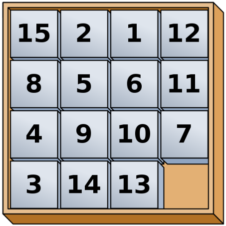
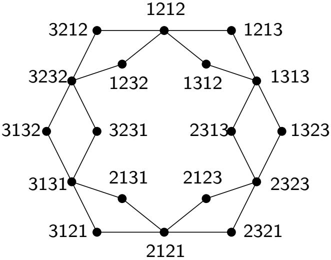
Reconfiguring vertex colourings of graphs





Alberta-Montana Combinatorics and Algorithms Days

June 3 - 5, 2022

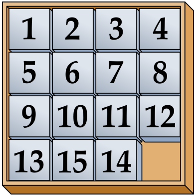
Michael Cavers

University of Calgary

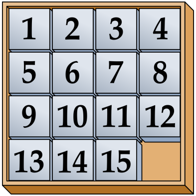
The 14-15 sliding puzzle

The 14-15 Puzzle:

Starting with a configuration with the 14 and 15 swapped (left image), use allowed moves to reconfigure the blocks from 1 to 15 (right image).



$\rightarrow \quad \cdots \quad ? \quad \cdots \quad \rightarrow$

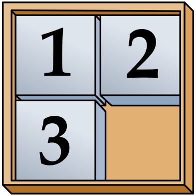


- **1870's:** Puzzlemaker Sam Loyd asked the public if anyone could solve the 14-15 sliding puzzle and offered a $1,000 prize (≈$22,000 today).
- He writes of how he "drove the entire world crazy":

> "People became infatuated with the puzzle and ludicrous tales are told of shopkeepers who neglected to open their stores; of a distinguished clergyman who stood under a street lamp all through a wintry night trying to recall the way he had performed the feat.
>
> The mysterious feature of the puzzle is that none seem able to remember the sequence of moves whereby they feel sure they have succeeded in solving the puzzle.
>
> Pilots are said to have wrecked their ships, and engineers rush their trains past stations."
>
> (Source: S. Loyd, Mathematical Puzzles of Sam Loyd: Selected and Edited by Martin Gardner, Dover, New York, 1959, pp. 19-20)

A simple example: The 3 sliding puzzle

- Consider the "**3 sliding puzzle.**"

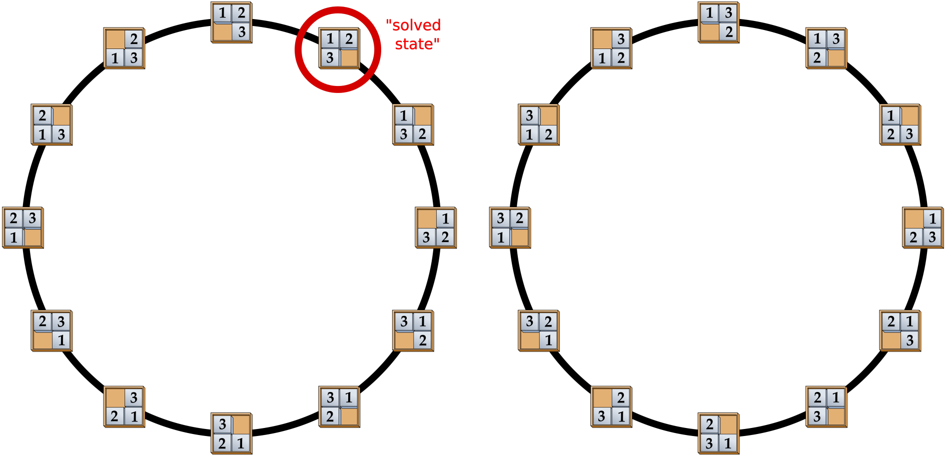


- **Vertices:** Each possible permutation is a vertex in the reconfiguration graph.
- **Edges:** If one state of the puzzle can be obtained from another by sliding a single square, then join the states by an edge.

Question: What is the reconfiguration graph for this problem?

Reconfiguration graph of the 3 sliding puzzle

If the blank is considered to be a 4, then there are $4! = 24$ possible permutations.



The 15 sliding puzzle: Approximately 21 trillion vertices of degrees 2, 3 and 4 with two components. The "15-14 configuration" and "solved state" are in different components!

Change Ringing

- The art of ringing tuned bells; common on church bells in English churches.
- Each bell has its own ringer.
- Due to weight, bells cannot be easily stopped/started giving limitations (e.g., we want to avoid ringing a bell twice in a row).

Some terminology:

- A piece of music for bells is called a **peal**.
  - In each row, every bell must ring once.
- Passing from one row to the next is a **change**.

The fundamental rule of change ringing:

- Each change must consist of transpositions of adjacent bells (i.e., a bell cannot move more than one position during a change).

More terminology:

- When only one bell is swapped with its neighbour it is a **plain change**.
- When all possible orderings of bells is used (once) we call it a **full peal** (or **extent**).

Example of call changes on eight bells

Showing bells being called "down" towards the lead

Direction of called bell — *Each swapped pair is shaded*

Row name	Row	Call	Strategy
Rounds	1 2 3 4 5 6 7 8	- *7 to 5*	Hunt 7, then 5, down: for ascending odds and descending evens at the back
	1 2 3 4 5 7 6 8	- *7 to 4*	
	1 2 3 4 7 5 6 8	- *7 to 3*	
	1 2 3 7 4 5 6 8	- *7 to 2*	
	1 2 7 3 4 5 6 8	- *5 to 3*	
	1 2 7 3 5 4 6 8	- *5 to 7*	
Whittingtons	1 2 7 5 3 4 6 8	- *5 to 2*	
	1 2 5 7 3 4 6 8	- *3 to 5*	Hunt 3,5,7, down: to intersperse odds and evens
	1 2 5 3 7 4 6 8	- *3 to 2*	
	1 2 3 5 7 4 6 8	- *3 to 1*	
	1 3 2 5 7 4 6 8	- *5 to 3*	
	1 3 5 2 7 4 6 8	- *7 to 5*	
Queens	1 3 5 7 2 4 6 8		
	⋮		

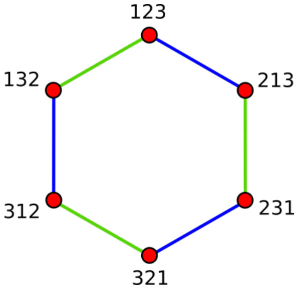
DS

(Source: Modification of work by Dougism)

Graph for 3 bells

Example

The graph for three bells using plain changes:

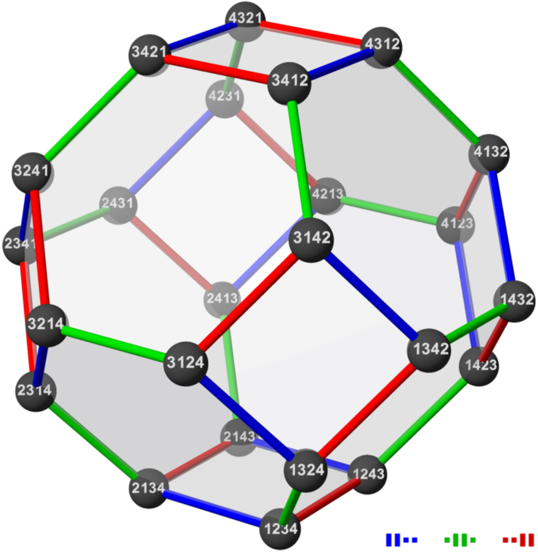


A full peal corresponds to a **Hamilton cycle**.

- The longest ever full peal was rung in England in 1963 on eight bells.
- It consisted of $8! = 40,320$ changes and took about 18 hours.

Graph for 4 bells using plain changes

A truncated octahedran or the S_4 permutohedron.



(Source: Tilman Piesk)

Background: Reconfiguration problems

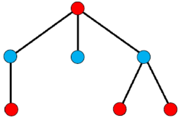
- A **reconfiguration problem** asks if a feasible solution to a problem can be transformed to another through a sequence of allowable steps.
- Feasibility must be maintained at each step.
- Reconfiguration problems have been studied for various graph problems:
  - colouring
  - dominating sets
  - vertex covers
  - cliques
  - independent sets

The chromatic number of a graph

$\chi(G)$ is the minimum number of colors required to colour the vertices of G so that no two adjacent vertices share the same colour.

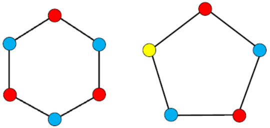
a path graph

$$\chi(P_n) = 2$$



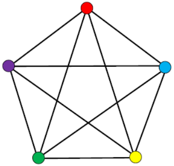
a tree graph

$$\chi(T) = 2$$



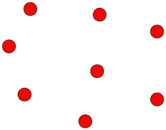
a cycle graph

$$\chi(C_n) = \begin{cases} 2, & \text{if } n \text{ even} \\ 3, & \text{if } n \text{ odd} \end{cases}$$



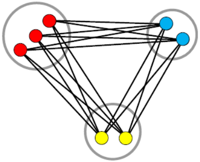
a complete graph

$$\chi(K_n) = n$$



an empty graph

$$\chi(\overline{K_n}) = 1$$



a complete multipartite graph

$$\chi(K_{a_1^{j_1}, a_2^{j_2}, \ldots, a_r^{j_r}}) = \#\text{ of parts}$$

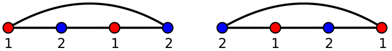
Definition: k-colourings of a graph

Given a graph $G = (V, E)$ and a positive integer k, a $\boxed{k\text{-colouring}}$ of G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ for which $f(x) \neq f(y)$ for any $xy \in E(G)$.

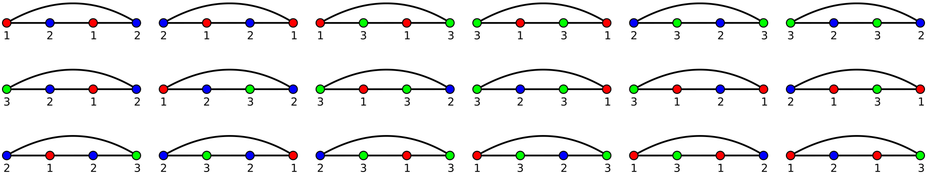
That is, a $\boxed{k\text{-colouring}}$ is an assignment of one of k possible labels (called colors) to the vertices of a graph such that no two adjacent vertices share the same color.

Example: k-colourings of the cycle C_4

There are two 2-colourings of the cycle C_4.



There are eighteen 3-colourings of the cycle C_4.



The k-colouring graph (i.e., reconfiguration graph of vertex colourings)

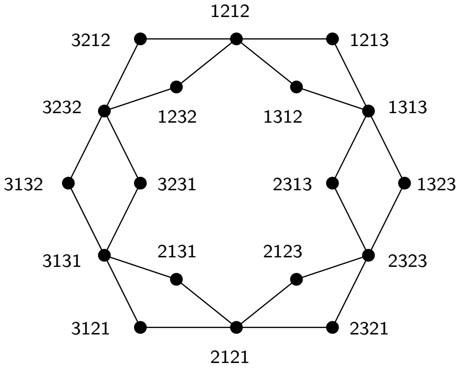
Definition: The k-colouring graph

The $\boxed{k\text{-colouring graph of } G}$ is denoted by $\mathcal{C}_k(G)$:

- **vertices**: all proper k-colourings of G
- **edges**: two k-colourings of G are adjacent if they differ on exactly one vertex of G

When $G = C_4$ then $\mathcal{C}_2(G)$ is $2K_1$.

Example: The 3-colouring graph of C_4, i.e., $\mathcal{C}_3(C_4)$)



Analysis of $\mathcal{C}_2(G)$ when G is bipartite

Remark (Cereceda, van den Heuvel and Johnson, 2008)

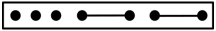
If G is bipartite with p isolated vertices and q other connected components, then

$$\mathcal{C}_2(G) \cong 2^q Q_p,$$

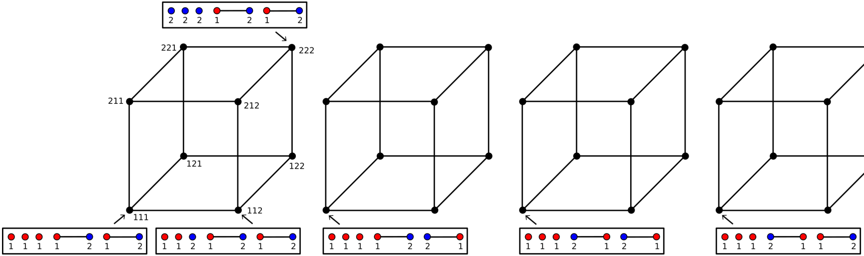
that is, $\mathcal{C}_2(G)$ has 2^q components each of which is a p-dimensional cube.

Example

Let G be a bipartite graph with $p = 3$ isolated vertices and $q = 2$ other components:



Then $\mathcal{C}_2(G)$ is $4Q_3$:



Question

Given a graph G, for which values of k is $\mathcal{C}_k(G)$ connected?

Definition: k-mixing

If $\mathcal{C}_k(G)$ is connected then we say that G is $\boxed{\textbf{k-mixing}}$.

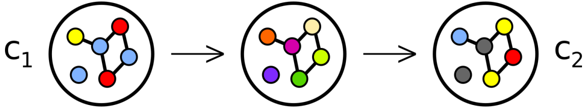
- The set of proper k-colourings of a graph G has been studied extensively by the Glauber dynamics Markov chain (i.e., a type of Markov Chain Monte Carlo algorithm) for k-colourings.
- Algorithms for random sampling of k-colourings and approximating the number of k-colourings arise from these Markov chains.
- For such a Markov chain to be rapidly mixing, we require the k-colouring graph of G to be connected.

The mixing number of a graph

Definition: mixing number

The $\boxed{\text{mixing number of } G}$ is the minimum integer $m_0(G)$ such that G is k-mixing whenever $k \geq m_0(G)$.

- Clearly: $m_0(G) \geq \chi(G)$.
- $m_0(G)$ is well-defined: When k is sufficiently large, we may find a path between a colouring c_1 and a colouring c_2 by recolouring each vertex of c_1 with unused colours not appearing yet and then recolouring them to their target colour.



- For a graph G, the k-colouring graph $\mathcal{C}_k(G)$ is connected when k is sufficiently large.

Questions

- Is there a graph G such that $\mathcal{C}_k(G)$ is connected for every $k \geq \chi(G)$?
- Is there an expression $\phi(\chi)$ in terms of the chromatic number χ such that for all graphs G and integers $k \geq \phi(\chi)$, the graph $\mathcal{C}_k(G)$ is connected?
- Is it possible for $\mathcal{C}_k(G)$ to be connected but $\mathcal{C}_{k+1}(G)$ to be disconnected?

Theorem (Cereceda, van den Heuvel and Johnson, 2008)

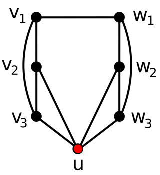
If $\chi(G) = 2$ then G is not 2-mixing (i.e., $\mathcal{C}_2(G)$ is not connected).
If $\chi(G) = 3$ then G is not 3-mixing (i.e., $\mathcal{C}_3(G)$ is not connected).

Their proof uses a lemma that describes how we can recognize that two 3-colourings of a graph are not connected by looking only at the colours of the vertices on a cycle.

Example: A graph that is k-mixing for all $k \geq \chi(G)$ (Cereceda et al., 2008)

Fix $m \geq 4$. Let H_m be obtained from two copies of K_{m-1} with vertex sets $\{v_1, \ldots, v_{m-1}\}$ and $\{w_1, \ldots, w_{m-1}\}$ by adding a vertex u and edges $v_1 w_1$ and $\{uv_i, uw_i \mid 2 \leq i \leq m-1\}$.

Below is H_4 which has chromatic number $\chi(H_4) = 4$ (generally $\chi(H_m) = m$).



Then H_m is k-mixing for every $k \geq m$, that is, $m_0(H_m) = \chi(H_m)$.

(See Theorem 7 and Lemmas 8 - 11 in their paper for details.)

Definition: frozen colourings

A k-colouring of G is **frozen** if each vertex of G is adjacent to at least one vertex of every other colour (a frozen k-colouring is an isolated vertex of $\mathcal{C}_k(G)$).

Example (Cereceda, van den Heuvel and Johnson, 2008)

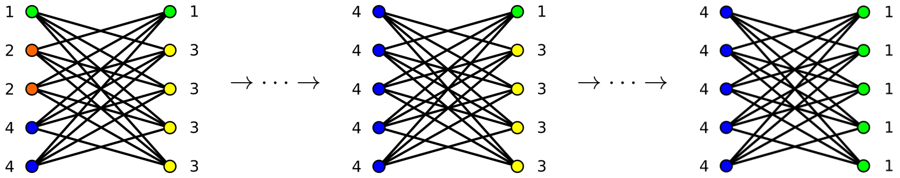
- Fix $m \geq 3$ and define the graph $L_m = K_{m,m} - mK_2$ obtained from the complete bipartite graph $K_{m,m}$ by deleting a perfect matching.
- Observe $\chi(L_m) = 2$.

- The 5-colouring of L_5 shown is frozen (i.e., an isolated vertex in $\mathcal{C}_5(L_5)$).
- Thus, $\mathcal{C}_m(L_m)$ is disconnected but $\chi(L_m) = 2$.

Theorem (Cereceda, van den Heuvel and Johnson, 2008)

Fix $m \geq 4$. Then $\mathcal{C}_k(L_m)$ is connected if $3 \leq k \leq m-1$ or $k \geq m+1$, but $\mathcal{C}_k(L_m)$ is disconnected if $k = 2$ or $k = m$.

Outline for $3 \leq k \leq m-1$:



- For $3 \leq k \leq m-1$, consider a k-colouring of L_m.
- There is at least one colour that appears on more than one vertex of the left side, say it is blue.
- This colour cannot appear on the right side.
- Now recolour every vertex on the left side with blue one at a time, then choose a second colour, say green, and recolour all vertices on the right side green.
- This shows every colouring is has a path to a 2-colouring of L_m.
- It is easy to show all 2-colourings of L_m are connected in $\mathcal{C}_k(L_m)$.

Upper bounds on the mixing number

Theorem (Jerrum, 1995)

For any graph G,

$$m_0(G) \leq \Delta(G) + 2$$

where $\Delta(G)$ is the maximum degree of G.

Theorem (Dyer, Flaxman, Frieze and Vigoda, 2006)

For any graph G,

$$m_0(G) \leq \mathrm{col}(G) + 2$$

where $\mathrm{col}(G)$ is the colouring number of G.

Theorem (Cereceda, van den Heuvel and Johnson, 2008)

For any graph G,

$$m_0(G) \leq \mathrm{col}(G) + 1$$

where $\mathrm{col}(G)$ is the colouring number of G.

(Note: Cereceda et al. use an alternate definition of $\mathrm{col}(G)$ which is adjusted in this statement.)

Definition: colouring number

The **colouring number of G**, denoted by $\mathrm{col}(G)$, is $\mathrm{col}(G) = \max\{\delta(H) \mid H \subseteq G\} + 1$.

The colouring number

Definition: colouring number

$$\mathrm{col}(G) = \max\{\delta(H) \mid H \subseteq G\} + 1.$$

Example

- **Trees:** $\mathrm{col}(T) = 2$
  Every subgraph H is a forest and thus $\delta(H) = 1$ (unless H has an isolated vertex).
- **Cycles:** $\mathrm{col}(C_n) = 3$
  Every subgraph is either C_n with $\delta(H) = 2$, a forest with $\delta(H) = 1$, or has an isolated vertex with $\delta(H) = 0$.
- **Complete graphs:** $\mathrm{col}(K_n) = n$
- **Unicyclic graphs:** $\mathrm{col}(U_n) = 3$

Example

Using $\chi(G) \le m_0(G) \le \mathrm{col}(G) + 1$ and that G is not $\chi(G)$-mixing when $\chi(G) \in \{2, 3\}$:

- **Trees:** $m_0(T) = 3$ for a tree T on at least two vertices;
- **Complete graphs:** $m_0(K_n) = n + 1$ since $\mathcal{C}_n(K_n)$ is disconnected;
- **Cycles:** $m_0(C_n) = 4$ when n is odd and $3 \le m_0(C_n) \le 4$ when n is even.
  - Cereceda et al. (2008) show that C_4 is the only 3-mixing cycle.

Definition: Grundy colouring

A proper k-colouring of G using colours $\{1, 2, \ldots, k\}$ is a **Grundy colouring** if, for $1 \leq i \leq k$, every vertex with colour i is adjacent to vertices of colours less than i.

Definition: Grundy number

The maximum number of colours required among all Grundy colourings of G is the **Grundy number** of G, denoted by $\chi_g(G)$.

$\chi_g(G)$ represents the greatest number of colors in a greedy coloring of G.

Theorem (Bonamy and Bousquet, 2018)

For any graph G,

$$m_0(G) \leq \chi_g(G) + 1.$$

If $k \geq \chi_g(G) + 1$ then $\mathrm{diam}(\mathcal{C}_k(G)) \leq 4n\chi(G)$.

Question

Given a graph G, for which values of k is $\mathcal{C}_k(G)$ Hamiltonian?

- **Choo and MacGillivray (2011):**
  For a graph G, the k-colouring graph $\mathcal{C}_k(G)$ is Hamiltonian when k is sufficiently large.

Definition: Gray code number

The $\boxed{\textbf{Gray code number of } G}$ is the least integer $k_0(G)$ for which $\mathcal{C}_k(G)$ has a Hamilton cycle for all $k \geq k_0(H)$.

- Clearly, $k_0(G) \geq m_0(G)$.
- Recall that $m_0(G) \leq \mathrm{col}(G) + 1$.

Theorem (Choo and MacGillivray, 2011)

For any graph G,

$$k_0(G) \leq \mathrm{col}(G) + 2.$$

That is, for every $k \geq \mathrm{col}(G) + 2$, the graph $\mathcal{C}_k(G)$ is Hamiltonian.

$k_0(G)$ for some graphs

Choo and MacGillivray (2011) prove that

- **Complete graphs:** $k_0(K_n) = n + 1$ for $n \geq 2$.
  (Note: $\mathrm{col}(K_n) = n$.)
  Proof Idea: $\mathcal{C}_n(K_n)$ has $n!$ vertices and no edges.
  $\mathcal{C}_{n+1}(K_n)$ is isomorphic to $Cay(X : S_{n+1})$ where X is the generating set of transpositions $\{(1,2),(1,3),\ldots,(1,n+1)\}$. This Cayley graph is shown to be Hamiltonian by Slater (1978).

- **Cycles:** $k_0(C_n) = 4$ for $n \geq 3$.
  (Note: $\mathrm{col}(C_n) = 3$.)

- **Trees:** $k_0(T) = 3$ with one exception:
  If T is a star with an odd number of vertices (greater than one) then $k_0(T) = 4$.
  (Note: $\mathrm{col}(T) = 2$.)

Celaya, Choo, MacGillivray and Seyffarth (2016) prove that

- **Complete bipartite graphs:** $k_0(K_{\ell,r}) = 3$ when ℓ and r are both odd, and $k_0(K_{\ell,r}) = 4$ otherwise.

Bard (2014) proves that

- **Complete multipartite graphs:** Fix $a_1, \ldots, a_t \in \mathbb{N}$. If $k \geq 2t$, then $\mathcal{C}_k(K_{a_1,\ldots,a_t})$ is Hamiltonian; Bard also provided improvements in special cases.

$k_0(G)$ for some graphs

Cavers and Seyffarth (2019) prove that

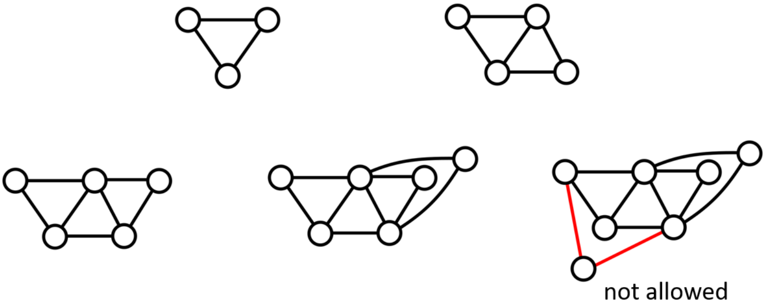
- **2-trees:** If H is a 2-tree then $k_0(H) = 4$, unless $H \cong T \vee \{u\}$ for some tree T and vertex u, where T is a star on at least three vertices or the bipartition of $V(T)$ has two even parts; in these cases, $k_0(H) = 5$.

Cavers, Seyffarth and Shaw (preprint) prove that

- **Unicyclic graphs:** If H is a unicylic graph then $k_0(H) = 4$, unless H has girth four and at least five vertices, in which case $k_0(H) = 3$.

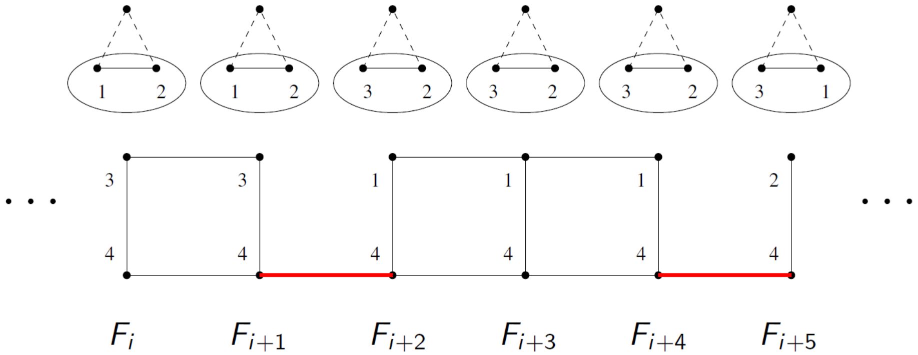
Definition: k-trees

A $\boxed{k\text{-tree}}$ is constructed recursively by starting with a complete graph on $k+1$ vertices and connecting each new vertex to an existing clique on k vertiecs (hence a 1-tree is simply a connected acyclic graph). A vertex of degree k is called a $\boxed{\textbf{leaf}}$.



- Let H be a k-tree.
- Then $\chi(H) = k+1$ and $\mathcal{C}_{k+1}(H)$ is disconnected.
- Since $\mathrm{col}(H) = k+1$, we have $k+2 \leq k_0(H) \leq k+3$.
- When $k = 2$ this gives $4 \leq k_0(H) \leq 5$.

- Induction with base case K_3.
- Let H be a 2-tree (with at least four vertices) and u be a leaf.
- Apply induction hypothesis to $H' = H - u$ to get a Hamilton cycle in $\mathcal{C}_4(H')$, denote it by $f_0 f_2 \ldots f_{N-1} f_0$ (each f_i is a 4-colouring of H').
- Let $F_j \subseteq V(\mathcal{C}_4(H))$ be the set of 4-colourings of H that agree with f_j on $V(H')$.
- Then $H[F_j] \cong K_2$ and $\{F_0, F_1, \ldots, F_{N-1}\}$ partitions the vertices of $\mathcal{C}_4(H)$.



(Source: Image by K. Seyffarth.)

Theorem (Cavers and Seyffarth, 2019)

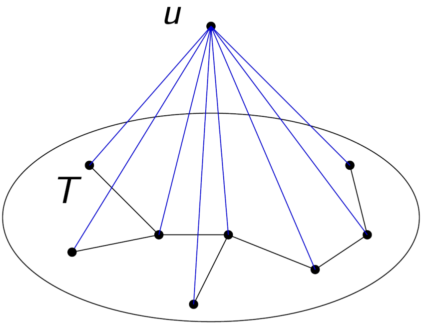
If H is a 2-tree then $k_0(H) = 4$, unless $H \cong T \vee \{u\}$ for some tree T and vertex u, where T is a star on at least three vertices or the bipartition of $V(T)$ has two even parts; in these cases, $k_0(H) = 5$.

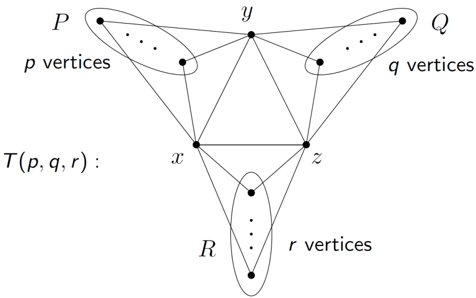
Proof Outline.

- We first characterize 2-trees of diameter two.
- We next determine the diameter two 2-trees H for which $\mathcal{C}_4(H)$ is not Hamiltonian.
- Finally, we show that for every 2-tree H with diameter at least three, the graph $\mathcal{C}_4(H)$ is Hamiltonian. To accomplish this, we specify a recursive procedure for constructing 2-trees of diameter at least three.

Lemma (Cavers and Seyffarth, 2019)

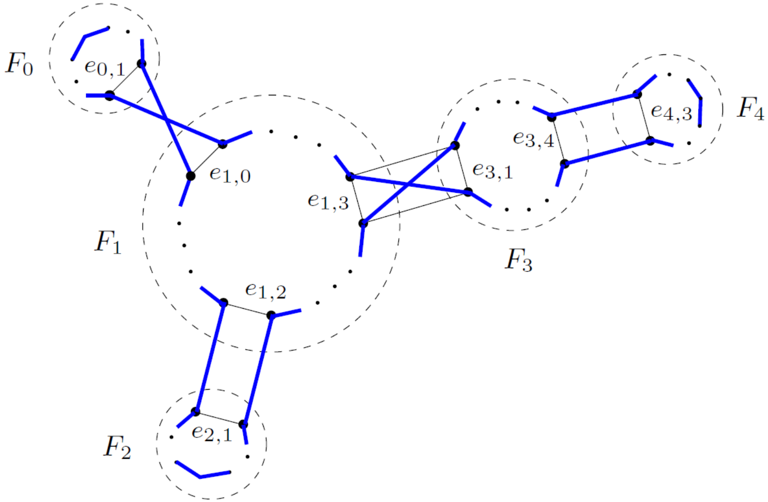
A graph H is a 2-tree of diameter two if and only if H has a dominating vertex or $H \cong T(p, q, r)$ for $p, q, r > 0$.





It turns out for $p, q, r > 0$, the graph $\mathcal{C}_4(T(p, q, r))$ has a Hamilton cycle.

- To prove $k_0(G) \leq \mathrm{col}(G) + 2$, Choo and MacGillivray introduced "C-graphs".
- We adapt this concept to construct Hamilton cycles in graphs.



Lemma (Cavers and Seyffarth, 2019)

Let G be a graph with vertex partition $\{F_0, F_1, \ldots, F_{N-1}\}$, and let T be a tree with $V(T) := \{f_0, f_1, \ldots, f_{N-1}\}$. Suppose there is a function, h, from a spanning subgraph of G to T such that $h(u) = f_i$ for all $u \in F_i$, $0 \leq i \leq N-1$. Furthermore, suppose that for each $f_i f_j \in E(T)$, $0 \leq i, j \leq N-1$, there exist edges $e_{i,j}$ in $G[F_i]$ and $e_{j,i}$ in $G[F_j]$ such that

(i) if $j \neq k$ and $f_i f_j, f_i f_k \in E(T)$, then $e_{i,j} \neq e_{i,k}$;

(ii) if $e_{i,j} = ac$ and $e_{j,i} = bd$, then $G[\{a, b, c, d\}]$ contains a 4-cycle;

(iii) $G[F_i]$ has a Hamilton cycle C_i such that

$$M_i := \{e_{i,j} \mid f_i f_j \in E(T)\} \subseteq E(C_i).$$

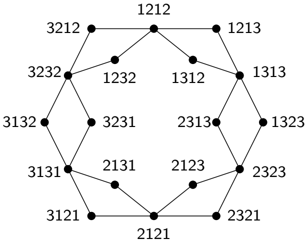
Then G has a Hamilton cycle C such that

$$\bigcup_{i=0}^{N-1} (E(C_i) \backslash M_i) \subseteq E(C).$$

Definition: Unicyclic graph

A graph G is called $\boxed{\textbf{unicyclic}}$ if G contains exactly one cycle.

- Let U_n be a unicyclic graph containing a cycle C_n of length n.
- Recall for cycles:
  $k_0(C_n) = 4$ for $n \geq 3$
  $m_0(C_n) = 4$ except when $n = 4$ in which case $m_0(C_4) = 3$.



Theorem (Cavers, Seyffarth and Shaw, preprint):
If H is a unicylic graph then $k_0(H) = 4$, unless H has girth four and at least five vertices, in which case $k_0(H) = 3$.

Question: Which graphs can be colouring graphs?

Given a graph H, do there exist G and k such that H is the k-colouring graph of G?

Beier, Fierson, Haas, Russell, Shavo (2016) proved that

- K_1 and P_2 are the only trees that are colouring graphs;
- every tree is a subgraph of a colouring graph (thus, there is no finite forbidden subgraph characterization of colouring graphs);
- C_3, C_4 and C_6 are the only cycles that are colouring graphs;
- If $\mathcal{C}_k(G)$ is a complete graph then it must be K_k;
- If $k > 1$ then $\mathcal{C}_k(G) = K_k$ if and only if $G = K_1$.

- Compute Gray code numbers of k-trees.
- Compute Gray code numbers of chordal graphs.
- Bard (2014):
  Is $K_{2,2,2}$ the only complete 3-partite graph whose 5-colouring graph is non-Hamiltonian?
- Bard (2014):
  If $\mathcal{C}_k(G)$ is Hamiltonian, is $\mathcal{C}_{k+1}(G)$ always Hamiltonian?
- Bonamy and Bousquet (2018):
  Given $r, k \in \mathbb{N}$, does there exist $c_{r,k}$ such that for any P_r-free graph G of order n that is k-mixing, the diameter of $\mathcal{C}_k(G)$ is at most $c_{r,k} \cdot n$?
- Bonsma and Cereceda (2009):
  Conjecture. For a graph G of order n and $k \geq \mathrm{col}(G) + 1$, $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^3)$.

References I

- S. Bard, *Grady code numbers of complete multipartite graphs*, Master's thesis, University of Victoria, 2014.
- J. Beier, J. Fierson, R. Haas, H. M. Russell, Heather, and K. Shavo, Classifying coloring graphs, *Discrete Math.*, **339** (2016), 2100–2112.
- M. Bonamy and N. Bousquet, Recoloring graphs via tree decompositions, *European J. Combin.*, **69** (2018), 200–213.
- P. Bonsma and L. Cereceda, Finding paths between graph colourings: PSPACE completeness and superpolynomial distances. *Theor. Comput. Sci.*, **410** (2009), 5215–5226.
- M. Cavers and K. Seyffarth, Reconfiguring vertex colourings of 2-trees, *Ars Math. Contemp.* **17** (2019), 653–698.
- M. Celaya, K. Choo, G. MacGillivray and K. Seyffarth, Reconfiguring k-colourings of complete bipartite graphs, *KyungPook Math. J.* **56** (2016), no. 3, 647–655.
- L. Cereceda, J. van den Heuvel and M. Johnson, Connectedness of the graph of vertex colourings, *Discrete Math.* **308** (2008), 913–919.
- L. Cereceda, J. van den Heuvel and M. Johnson, Mixing 3-colourings in bipartite graphs, *Eur. J. Comb.* **30** (2009), 1593–1606.

- L. Cereceda, J. van den Heuvel and M. Johnson, Finding paths between 3-colorings, *J. Graph Theory*, **67** (2011), 69–82.
- K. Choo, *The existence of grey codes for proper k-colourings of graphs*, Master's thesis, University of Victoria, 2002.
- K. Choo and G. MacGillivray, Gray code numbers for graphs, *Ars Math. Contemp.* **4** (2011), 125–139.
- M. Dyer, A. Flaxman, A. Frieze, E. Vigoda, Randomly coloring sparse random graphs with fewer colors than the maximum degree, *Random Structures Algorithms*, **29** (2006), 450–465.
- M. Jerrum, A very simple algorithm for estimating the number of k-colorings of a low-degree graph, *Random Structures Algorithms* **7** (1995), 157–165.
- C.M. Mynhardt and S. Nasserasr, Reconfiguration of colourings and dominating sets in graphs: a survey, in F. Chung, R. Graham, F. Hoffman,L. Hogben, Ron Mullin and Doug West (Eds.), *50 Years of Combinatorics, Graph Theory, and Computing*, Chapman and Hall/CRC Press, 2019.