hex etc – open problems

email hayward@ualberta.ca

AMCAD@BIRS June 3-5 2022

Gardner July '57 SciAm game of hex

JULY 1957 Vol 197#1 145-150 PLUS REPLIES/SOL'NS

MATHEMATICAL GAMES

Concerning the game of Hex, which may be played on the tiles of the bathroom floor

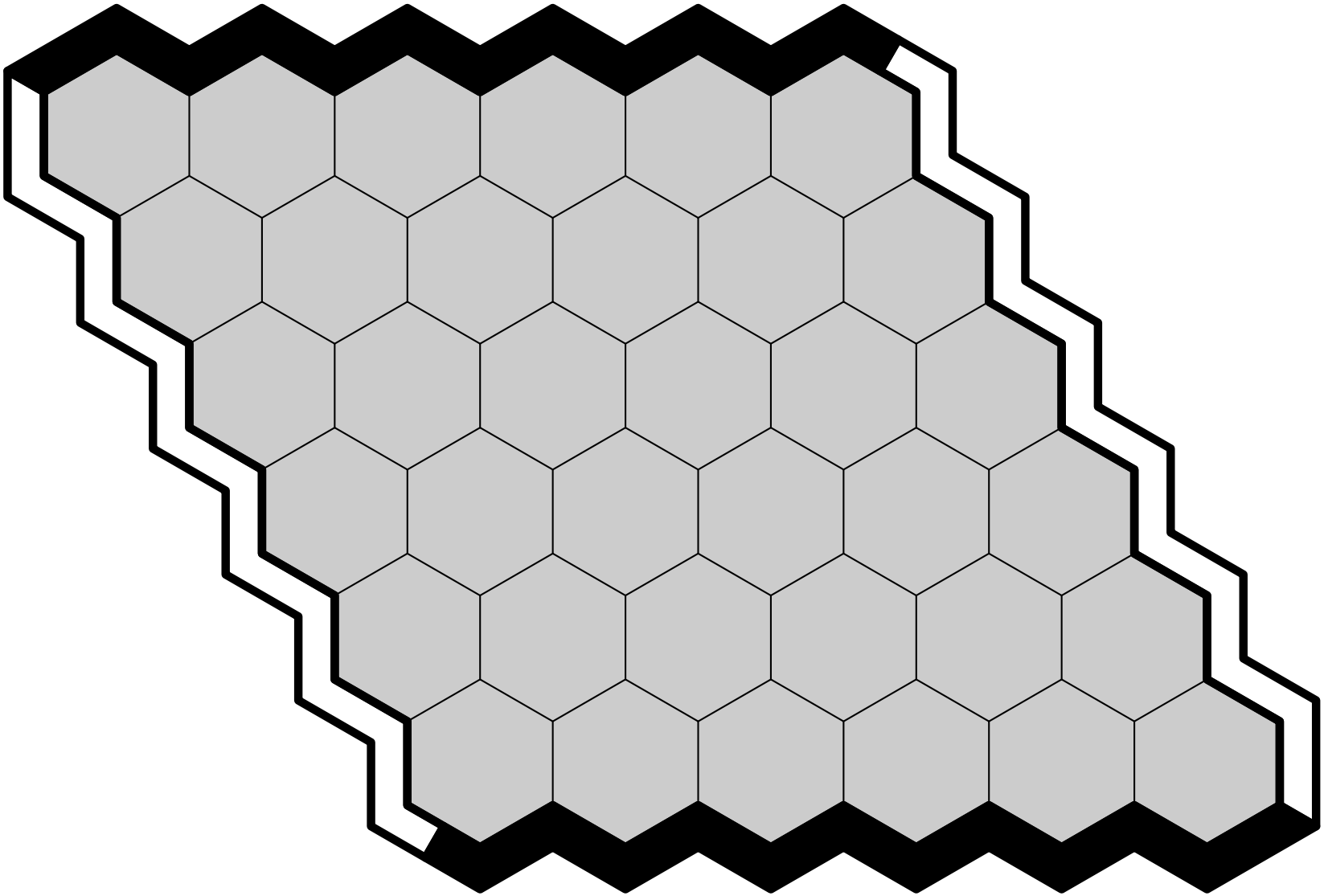
by Martin Gardner

It is something of an occasion these days when someone invents a mathematical game that is both new and interesting. Such a game is Hex, introduced 15 years ago at Niels Bohr's Institute for Theoretical Physics in Copenhagen. Hex may well become one of the most widely played and thoughtfully analyzed new mathematical games of the century. It swept the Scandinavian countries in the middle 1940s, and in 1949 it was taken up by game theorists in the U. S. Later Claude E. Shannon and Edward F. Moore of the Bell Telephone Laboratories designed and built an analogue computer capable of playing a moderately good game of Hex.

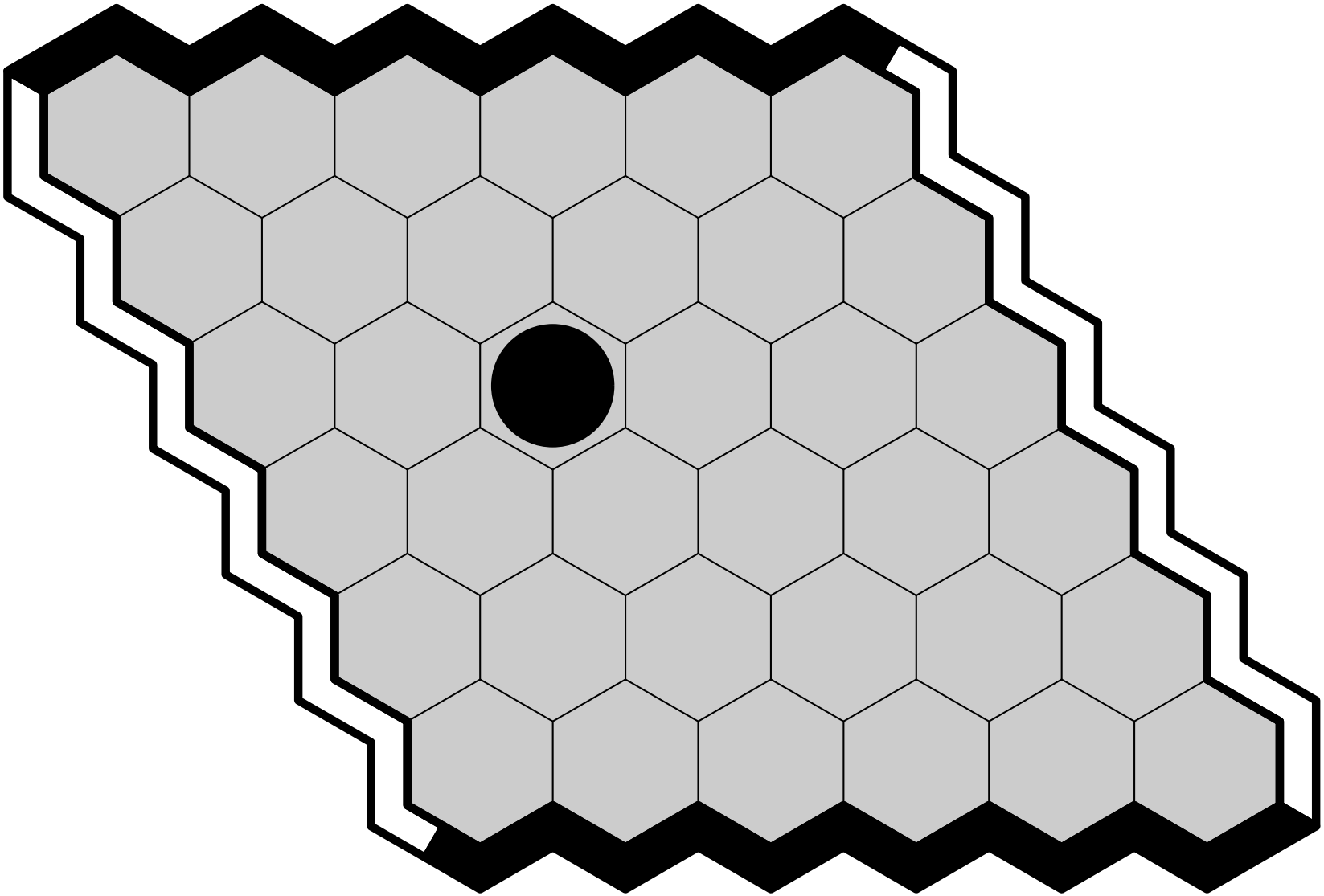
supply of white pieces. The players alternately place one of their pieces on any one of the hexagons, provided the hexagon is not already occupied by another piece. The objective of "black" is to complete an unbroken chain of black pieces between the two sides labeled "black." "White" tries to complete a similar chain of white pieces between the sides labeled "white."

The chain may freely twist and turn; an example of a winning chain is shown in the illustration at the bottom of the page. The players continue placing their pieces until one of them has made a complete chain. The game cannot end in a draw, because one player can block the other only by completing his own chain. These rules are simple, yet Hex is a game of surprising mathematical subtlety.

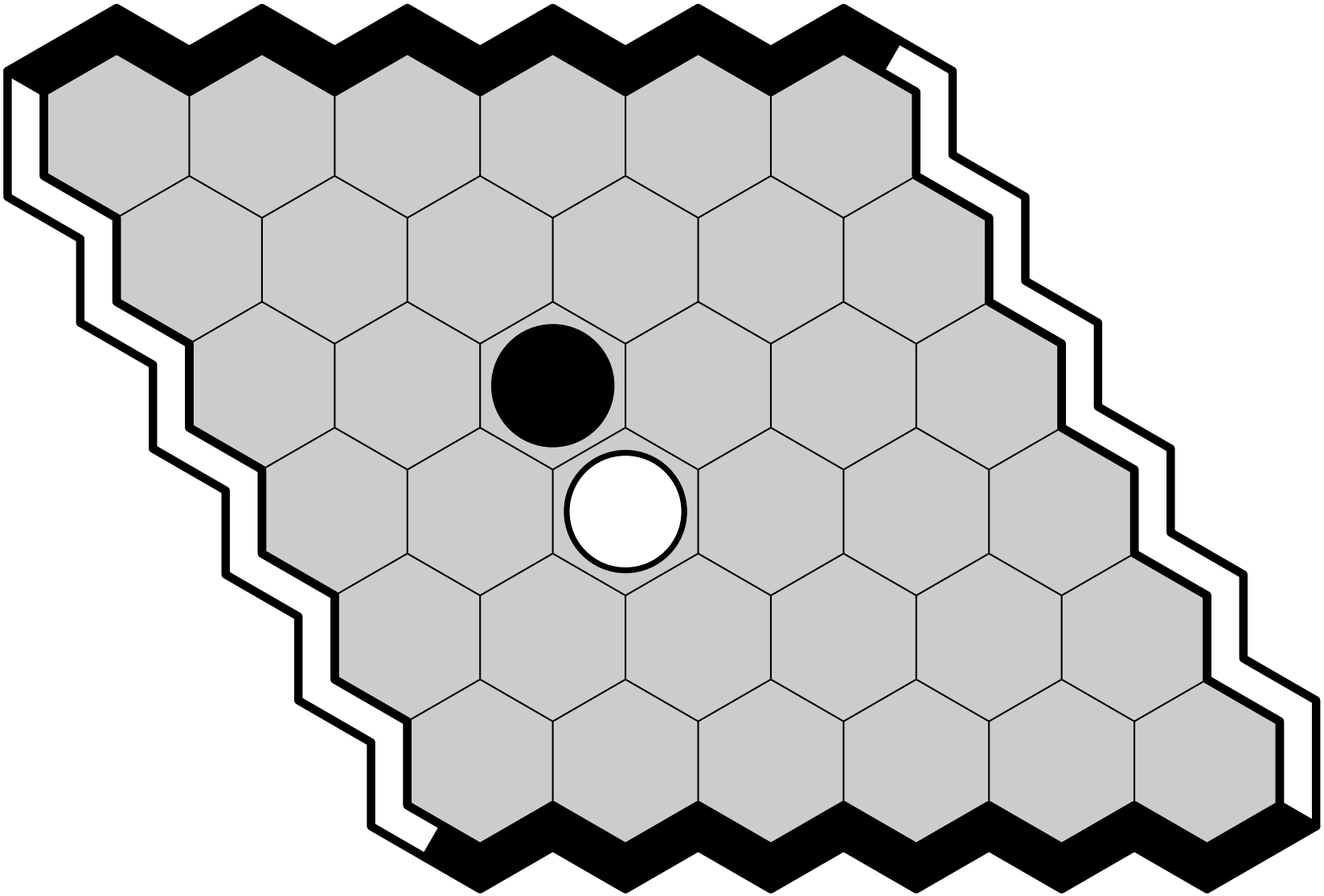
Gardner July '57 SciAm game of hex



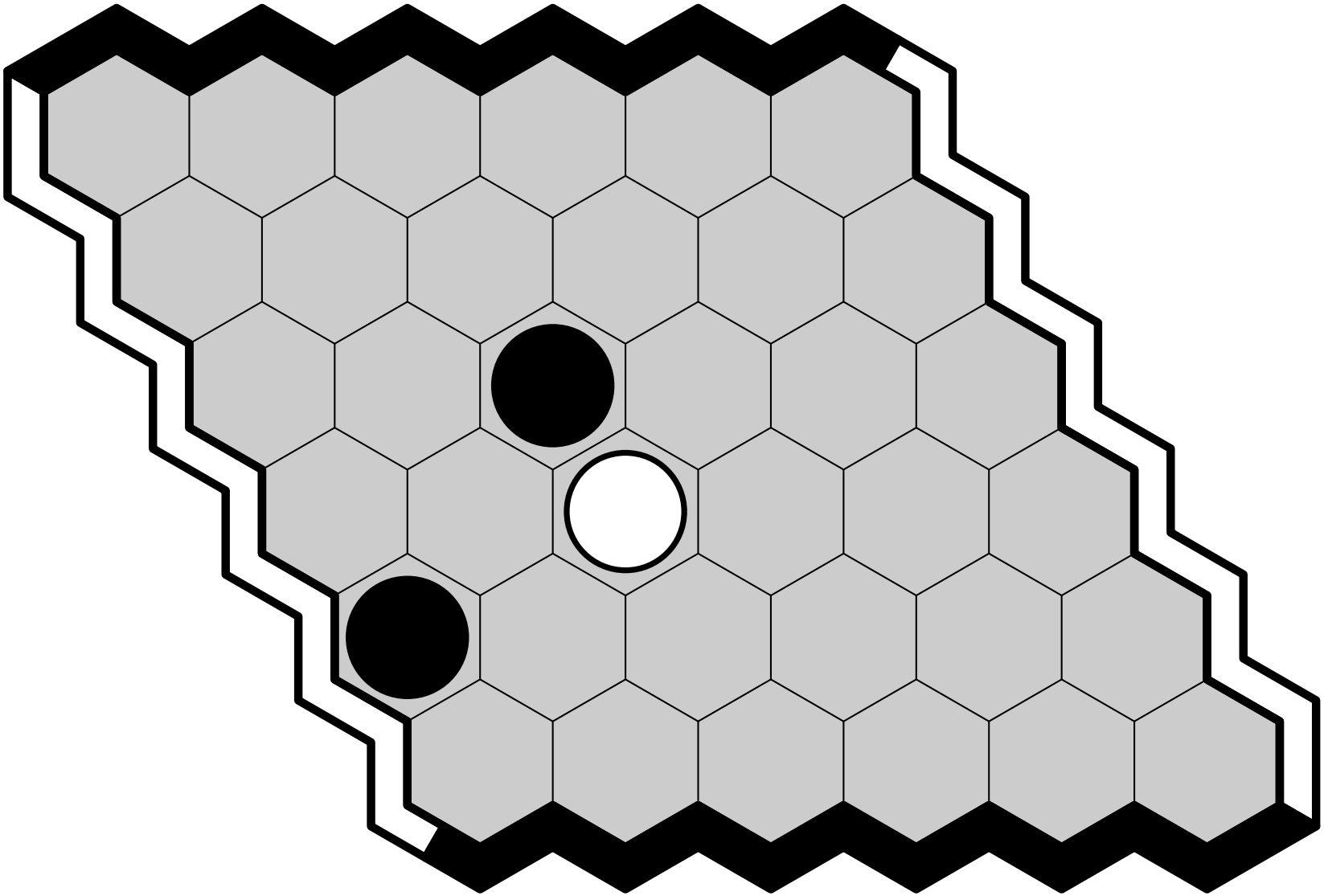
Gardner July '57 SciAm game of hex



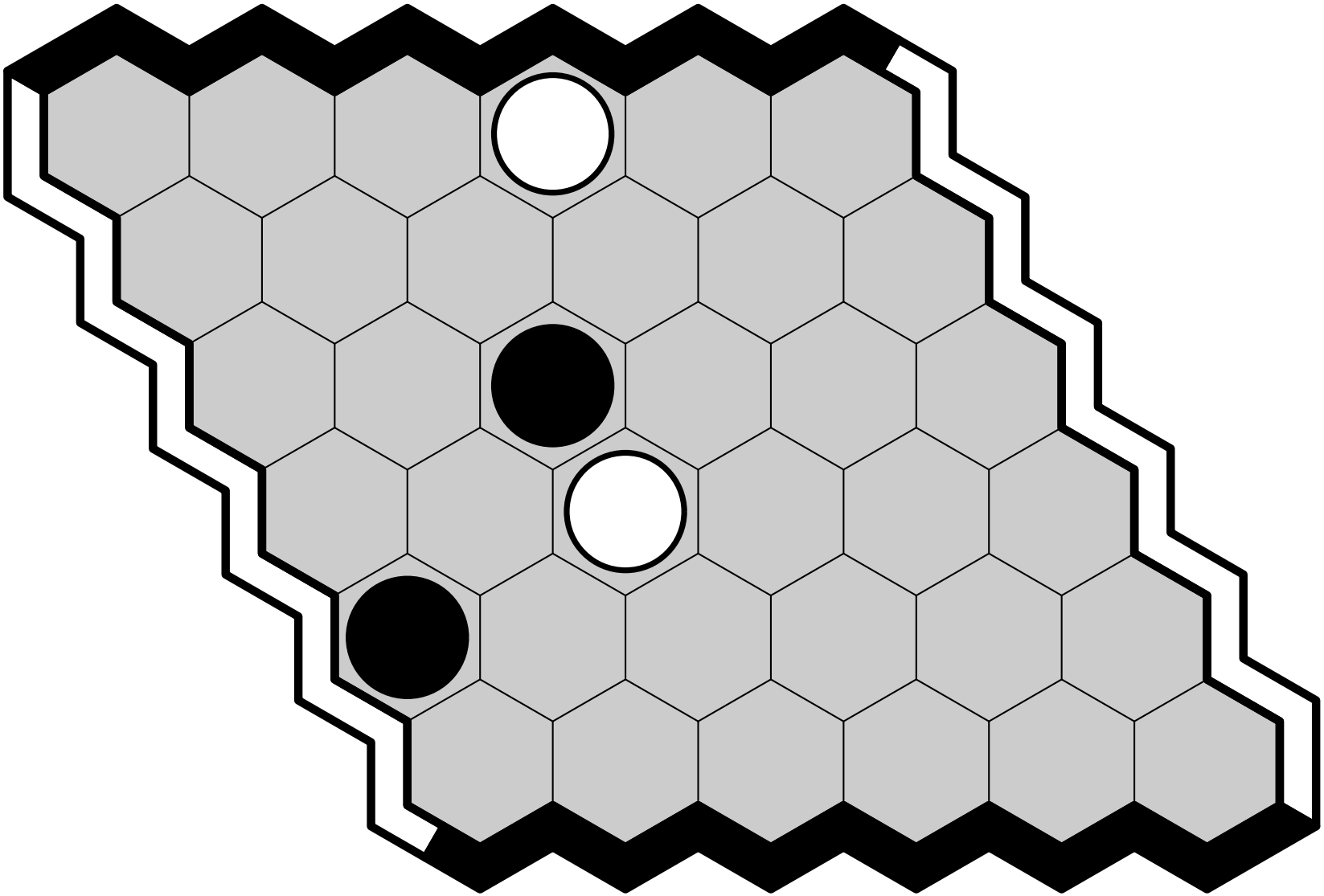
Gardner July '57 SciAm game of hex



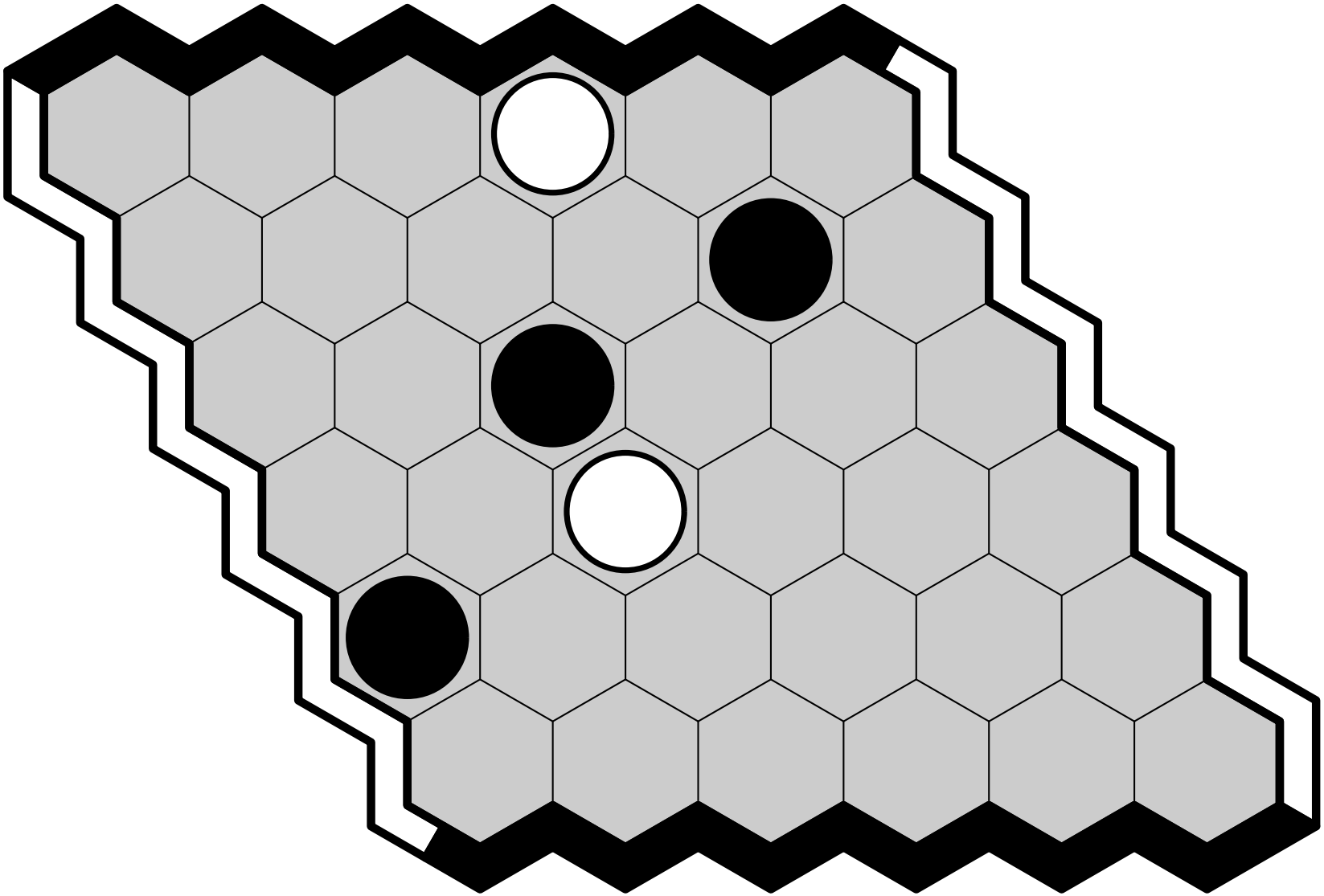
Gardner July '57 SciAm game of hex



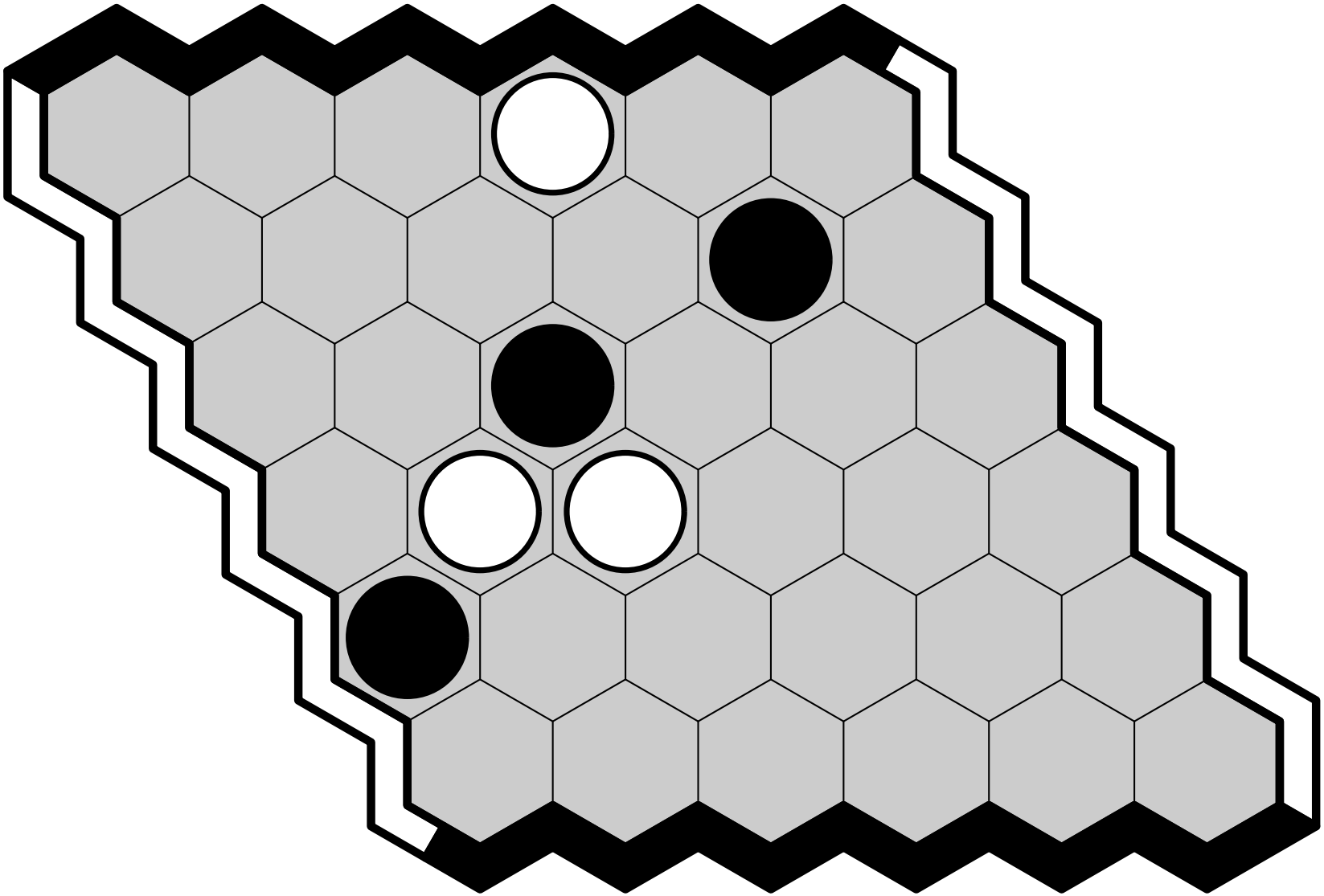
Gardner July '57 SciAm game of hex



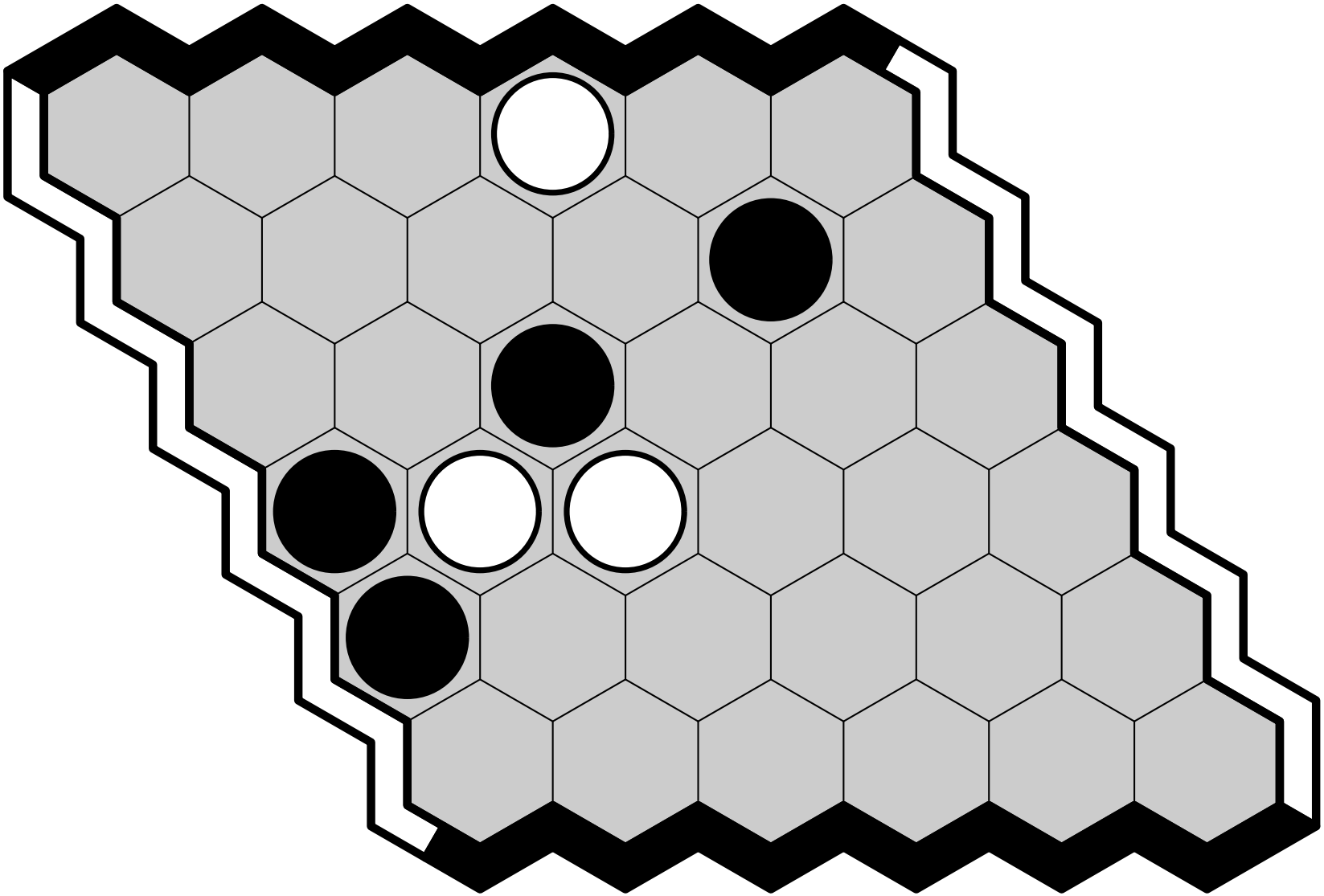
Gardner July '57 SciAm game of hex



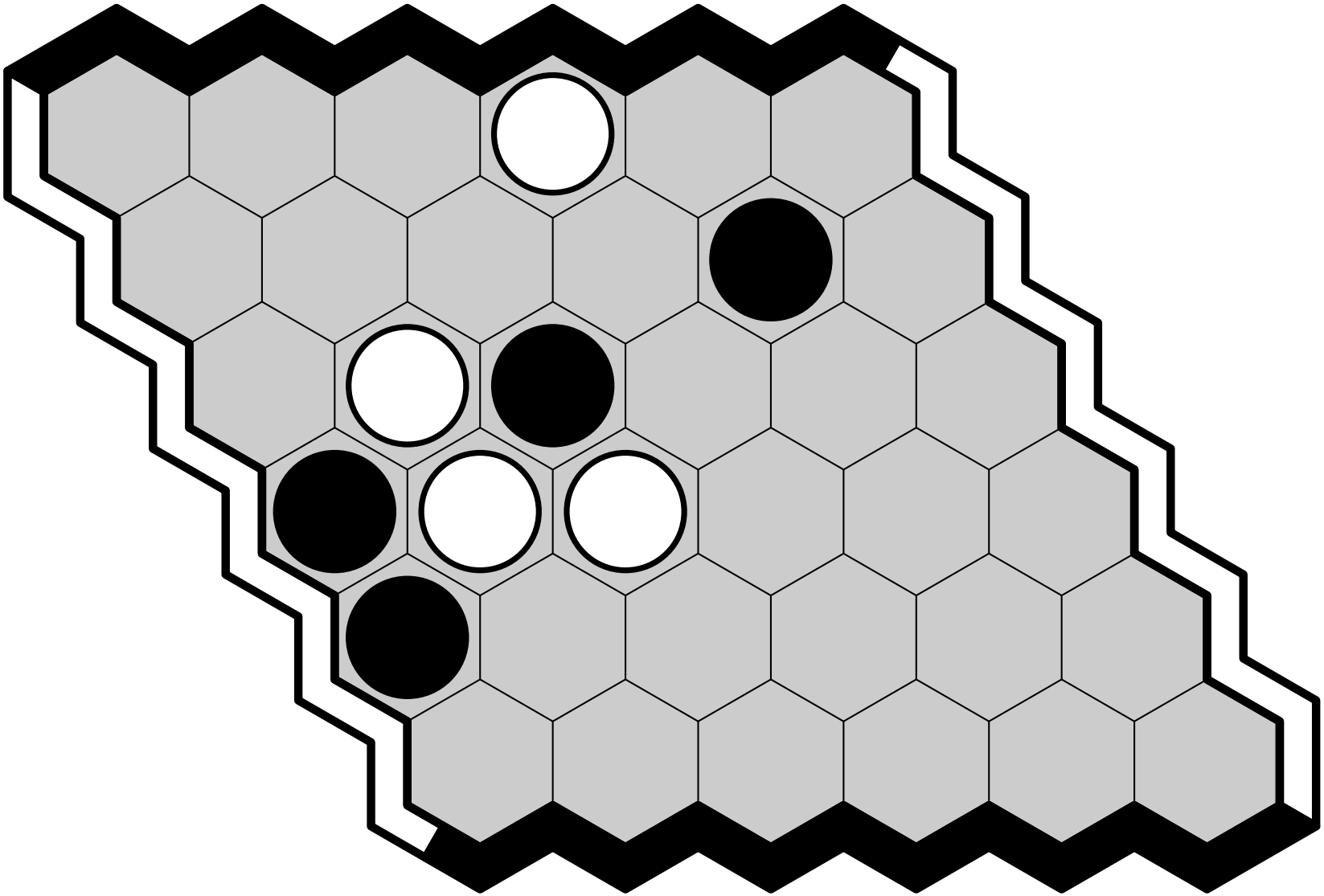
Gardner July '57 SciAm game of hex



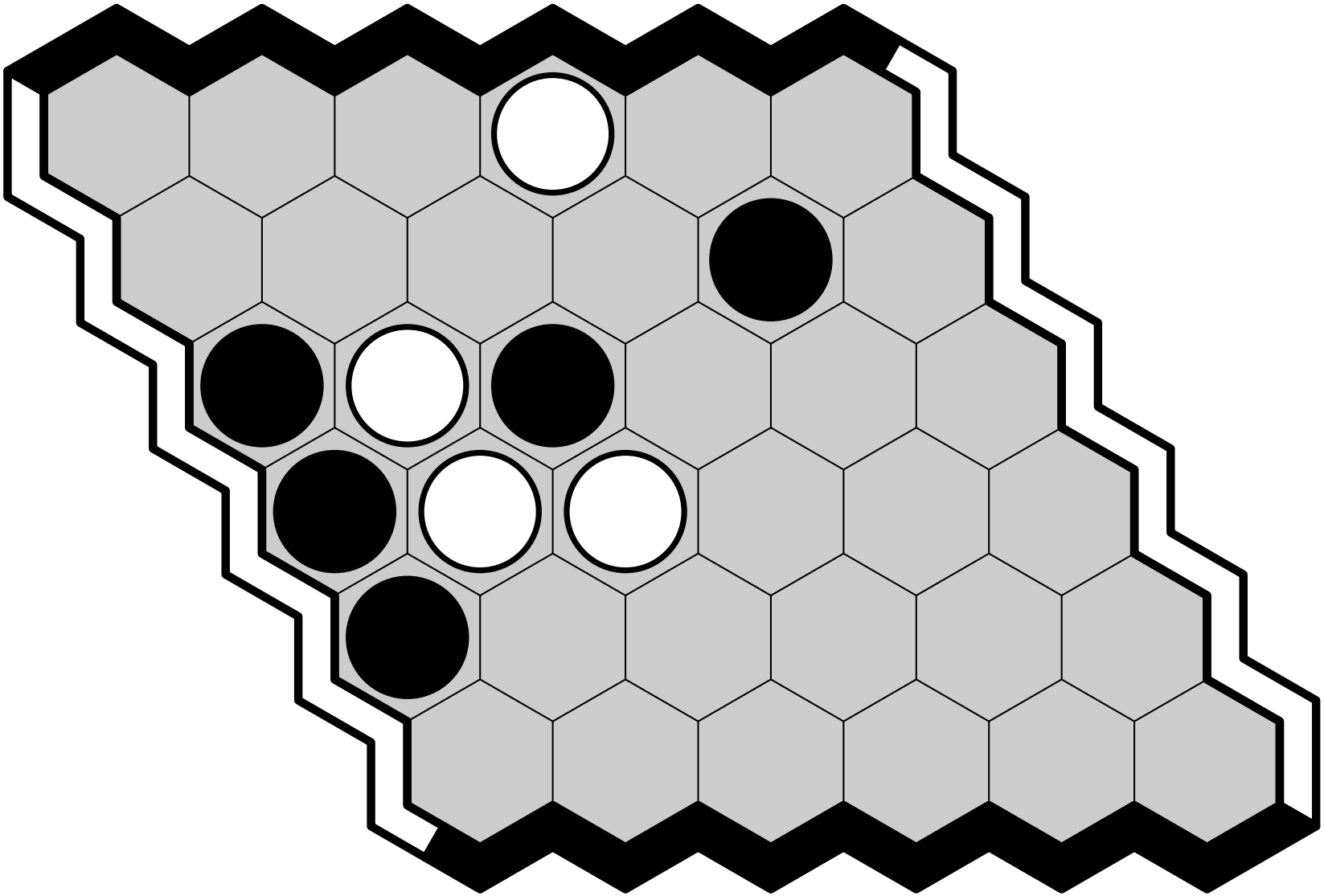
Gardner July '57 SciAm game of hex



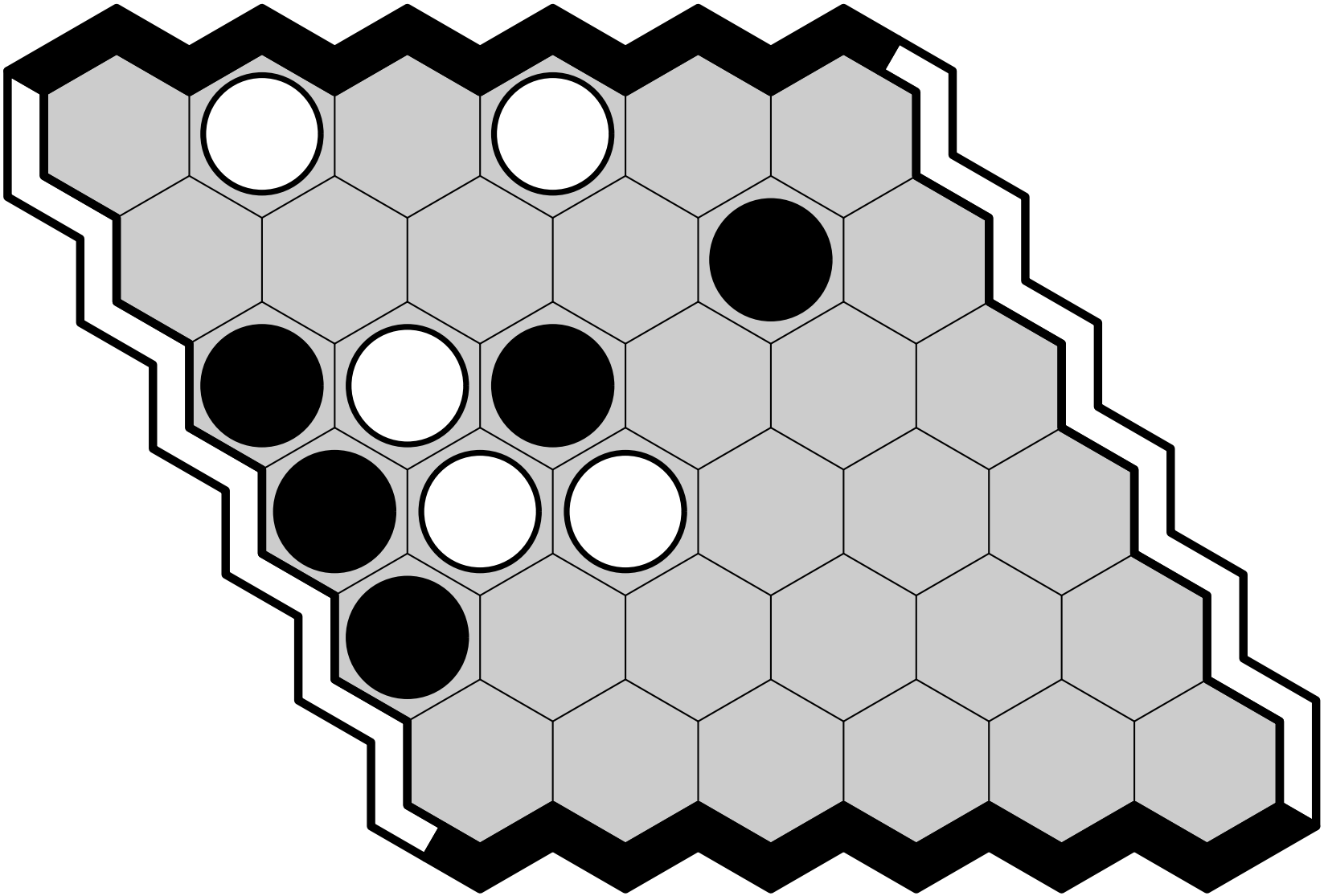
Gardner July '57 SciAm game of hex



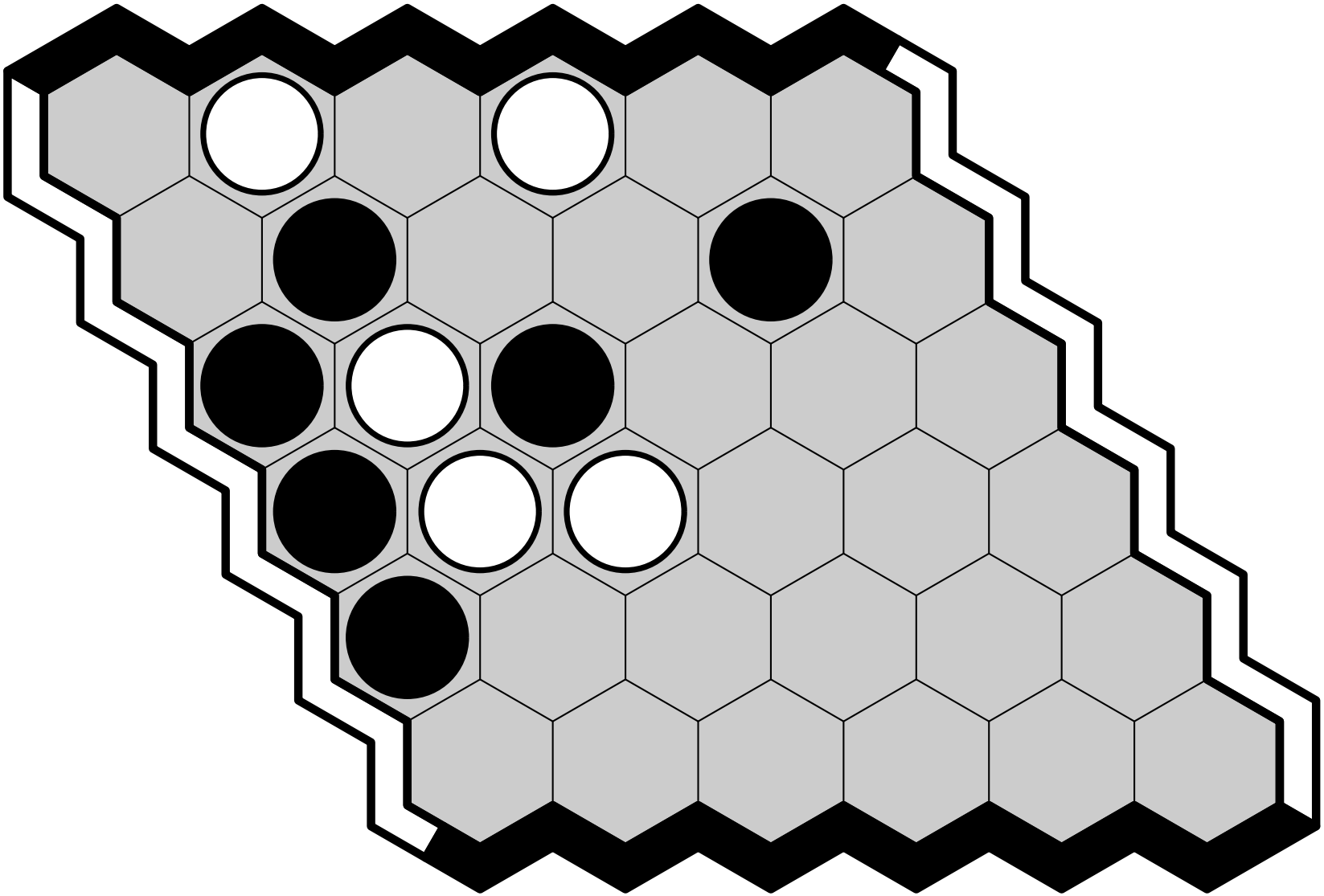
Gardner July '57 SciAm game of hex



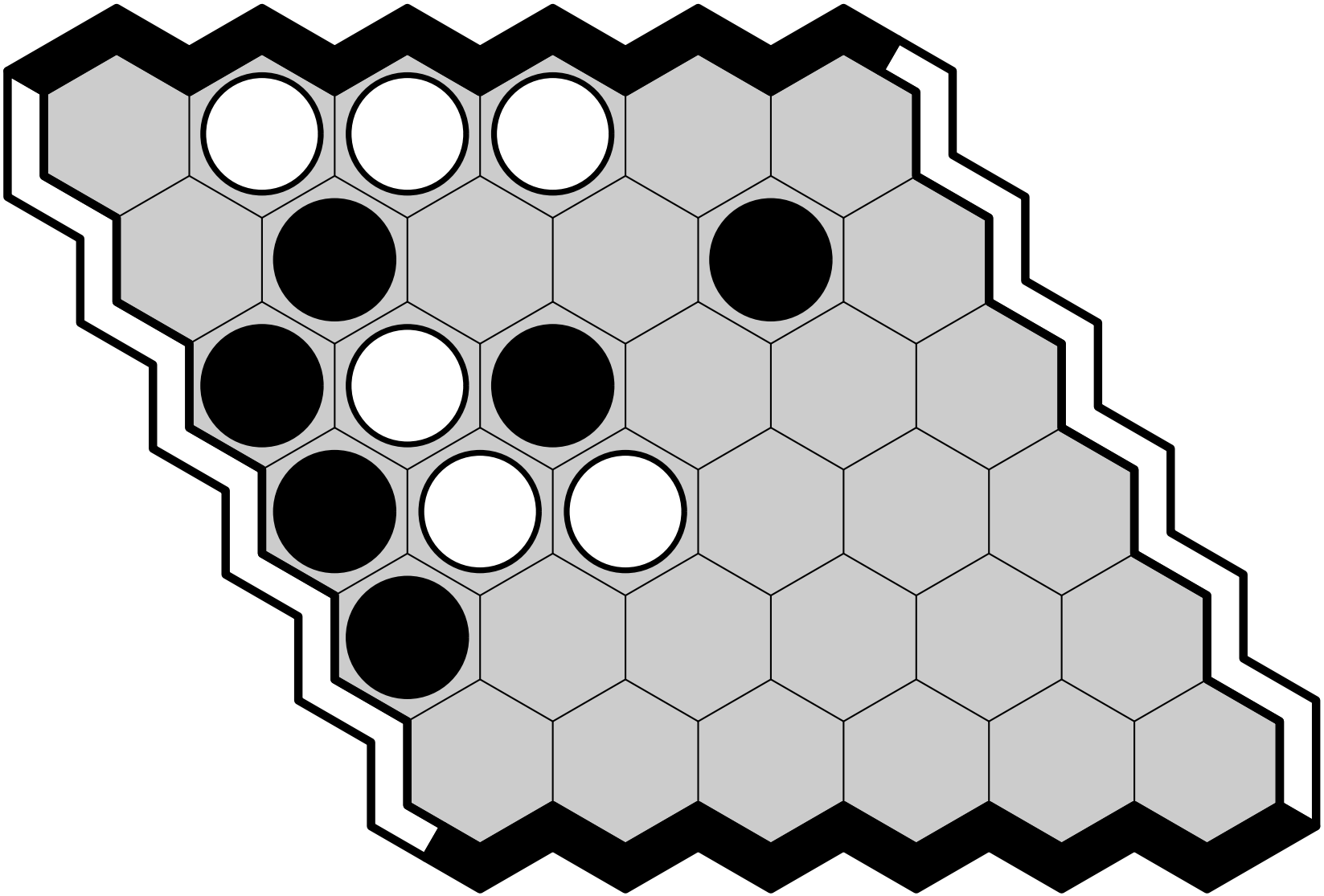
Gardner July '57 SciAm game of hex



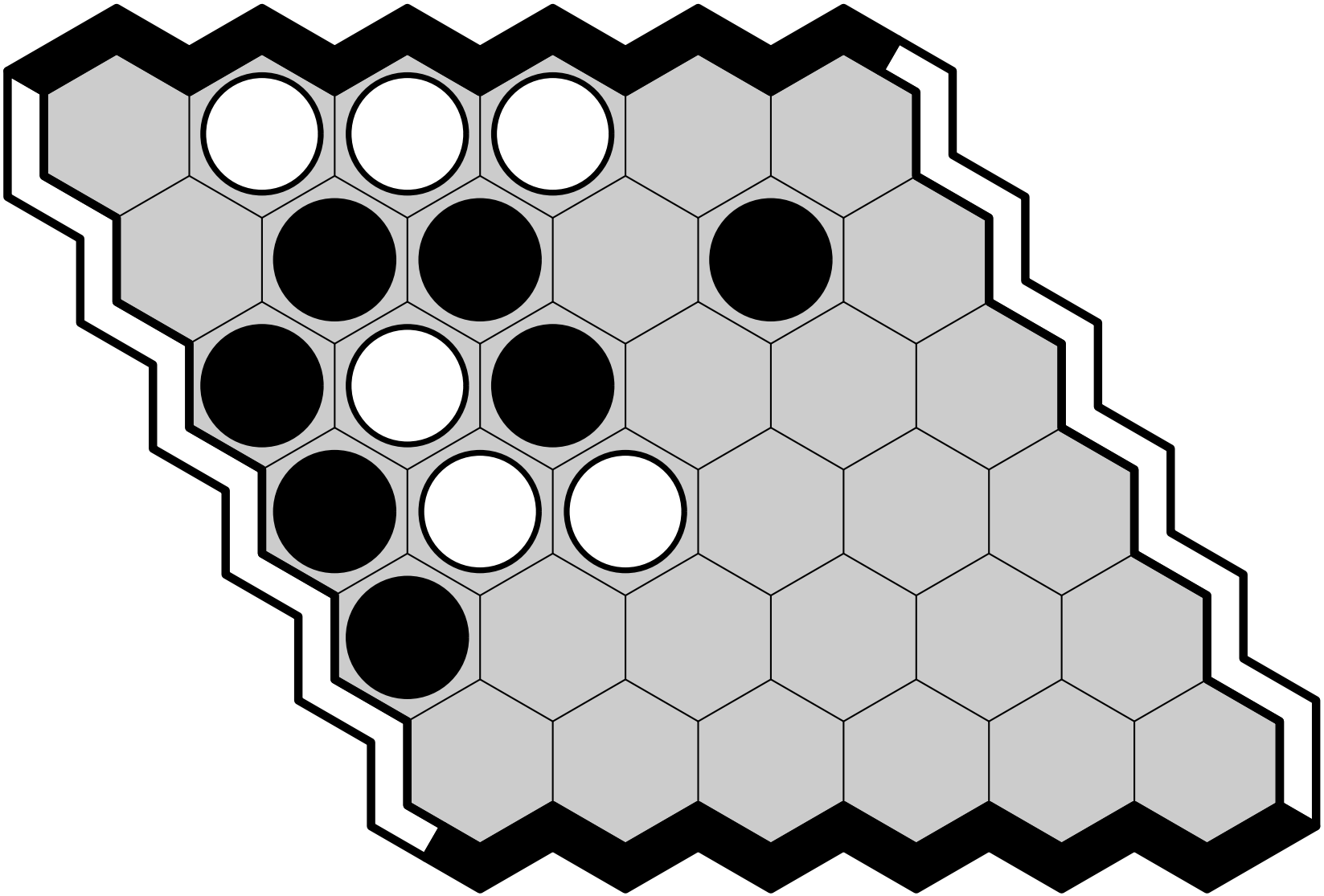
Gardner July '57 SciAm game of hex



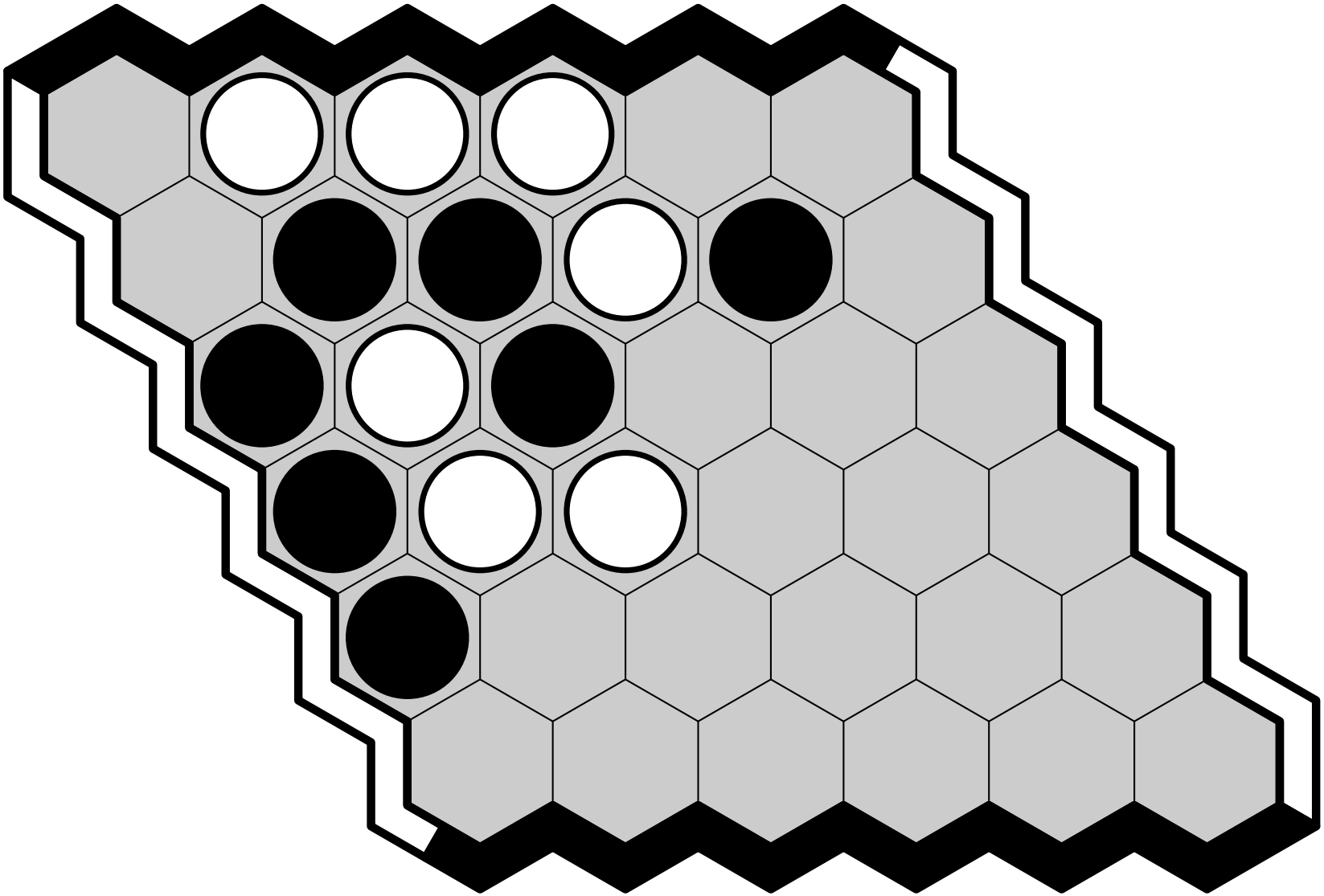
Gardner July '57 SciAm game of hex



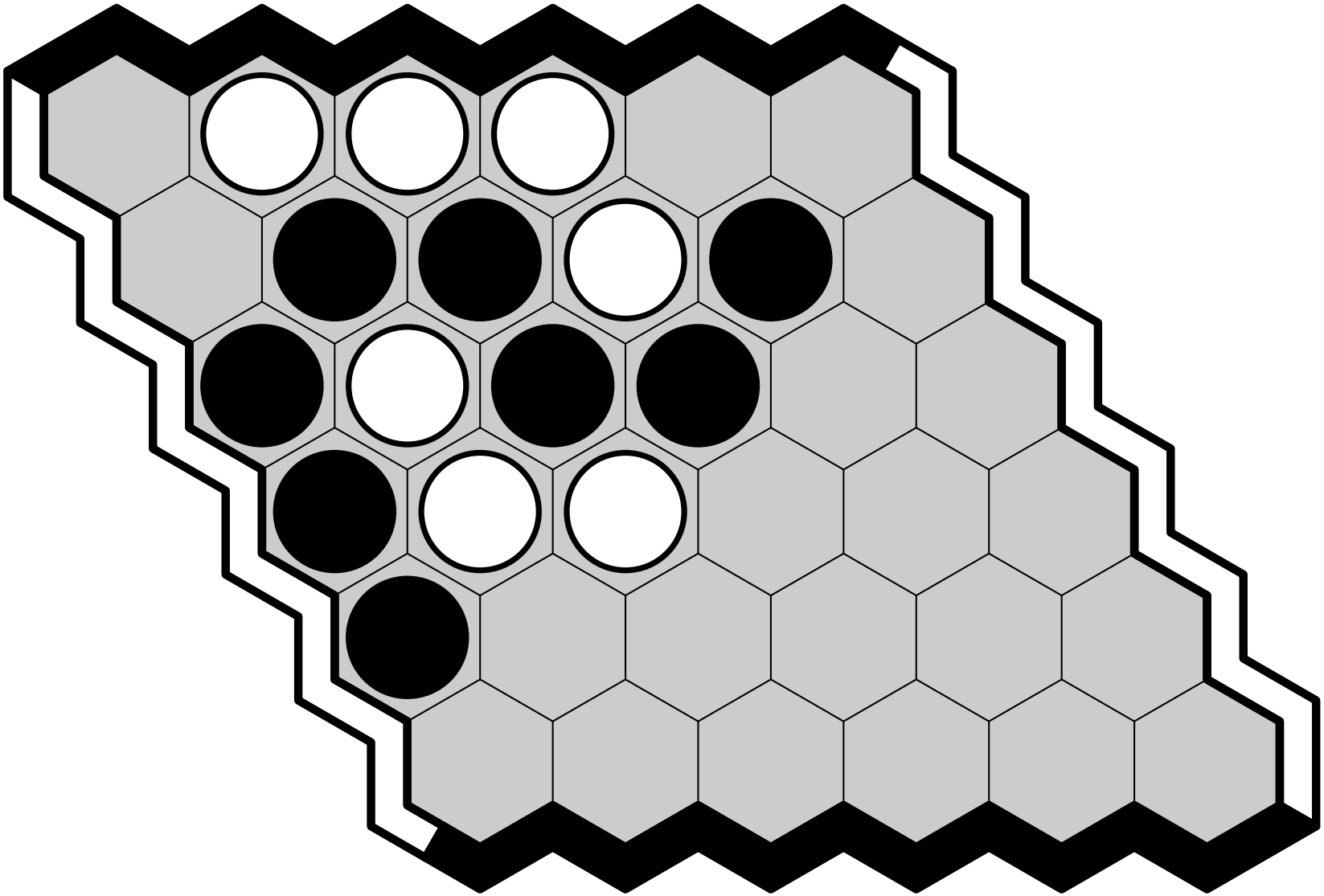
Gardner July '57 SciAm game of hex



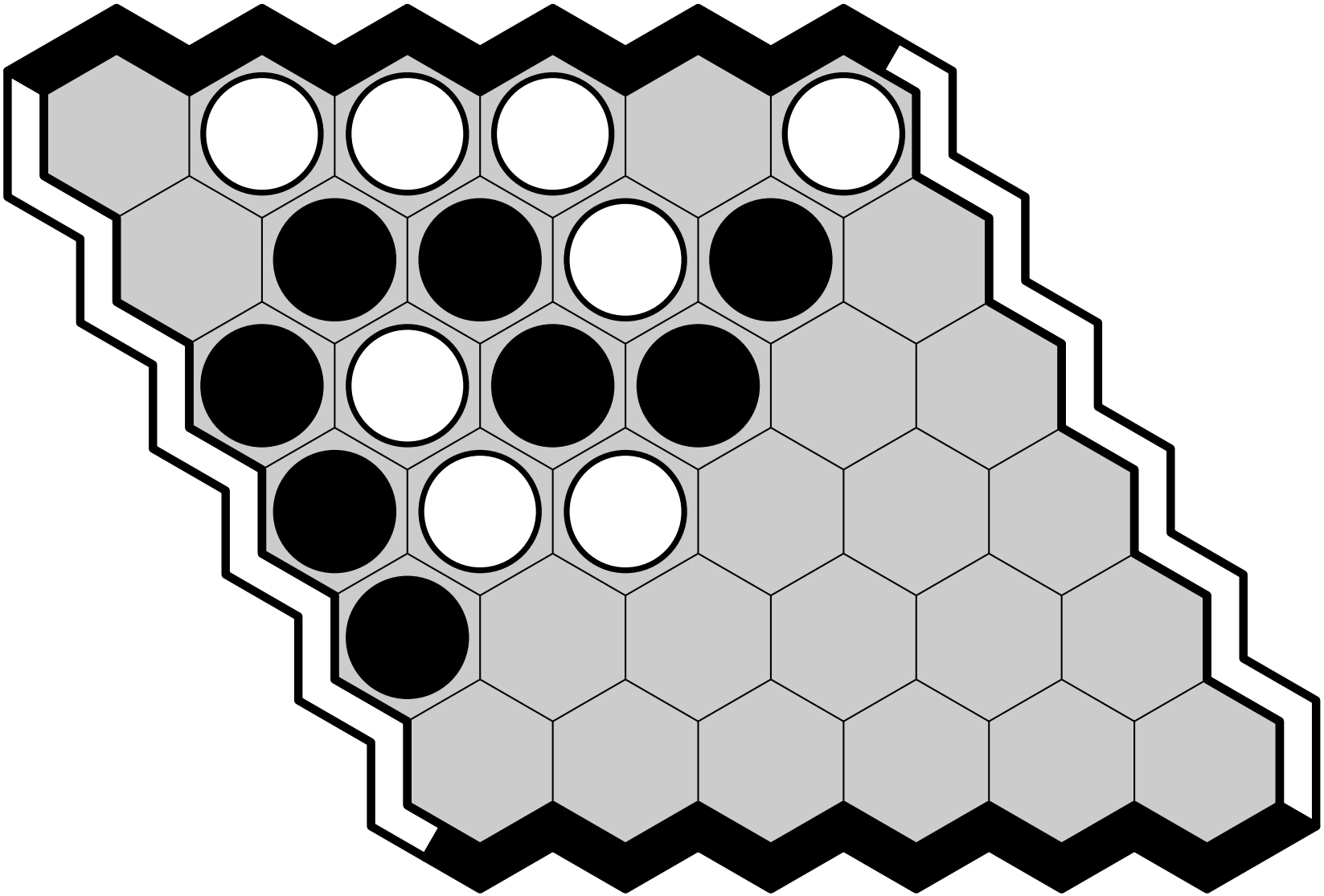
Gardner July '57 SciAm game of hex



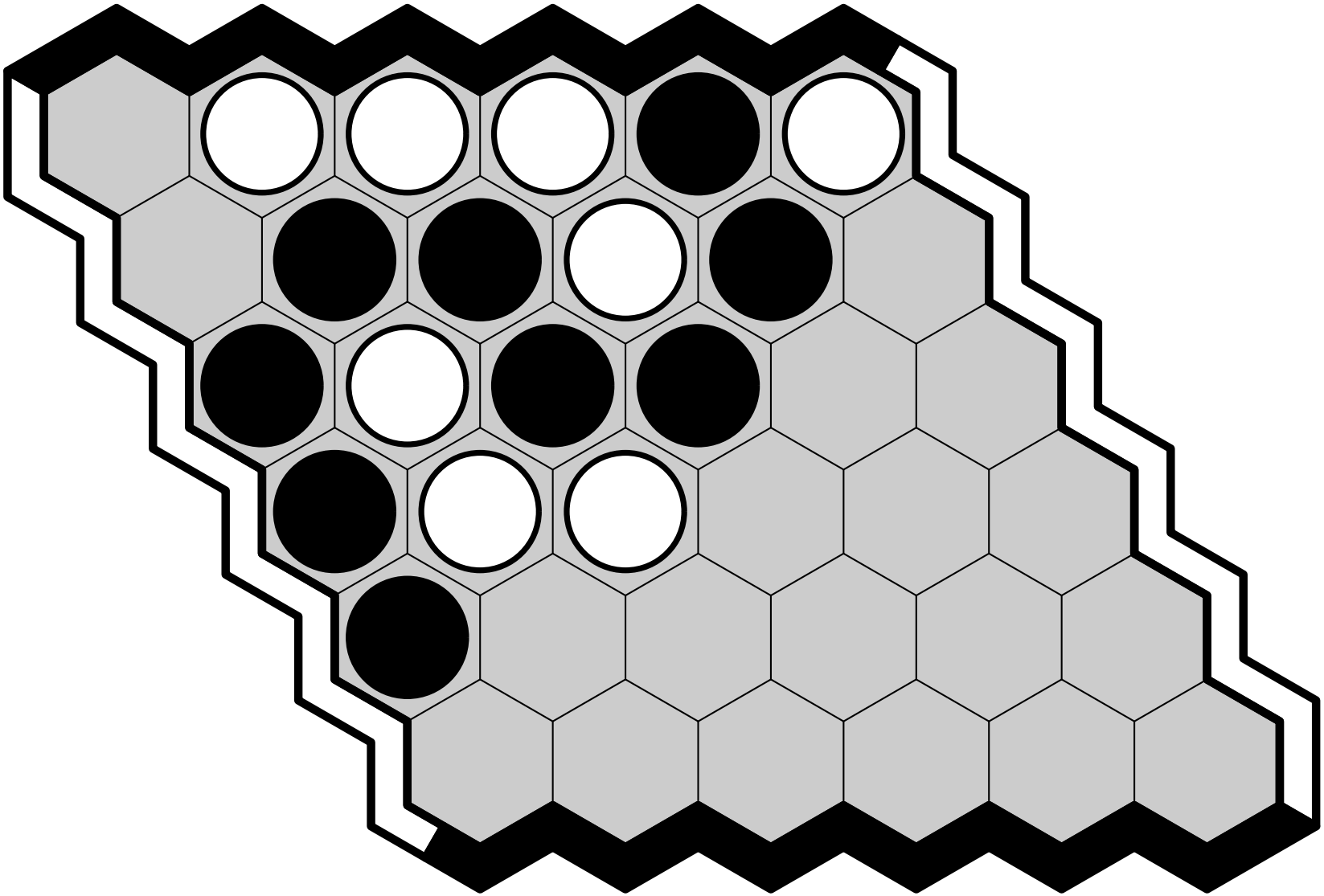
Gardner July '57 SciAm game of hex



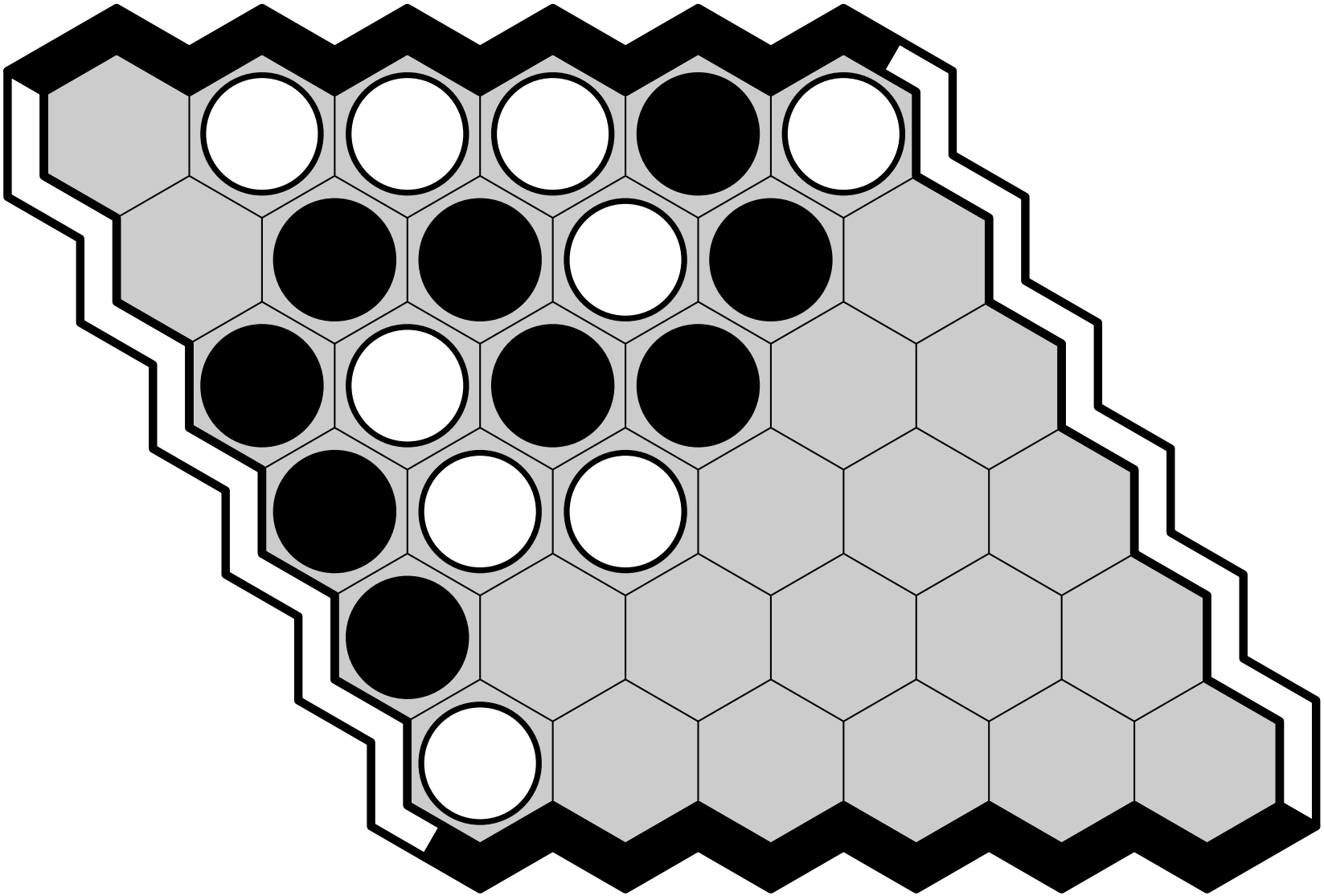
Gardner July '57 SciAm game of hex



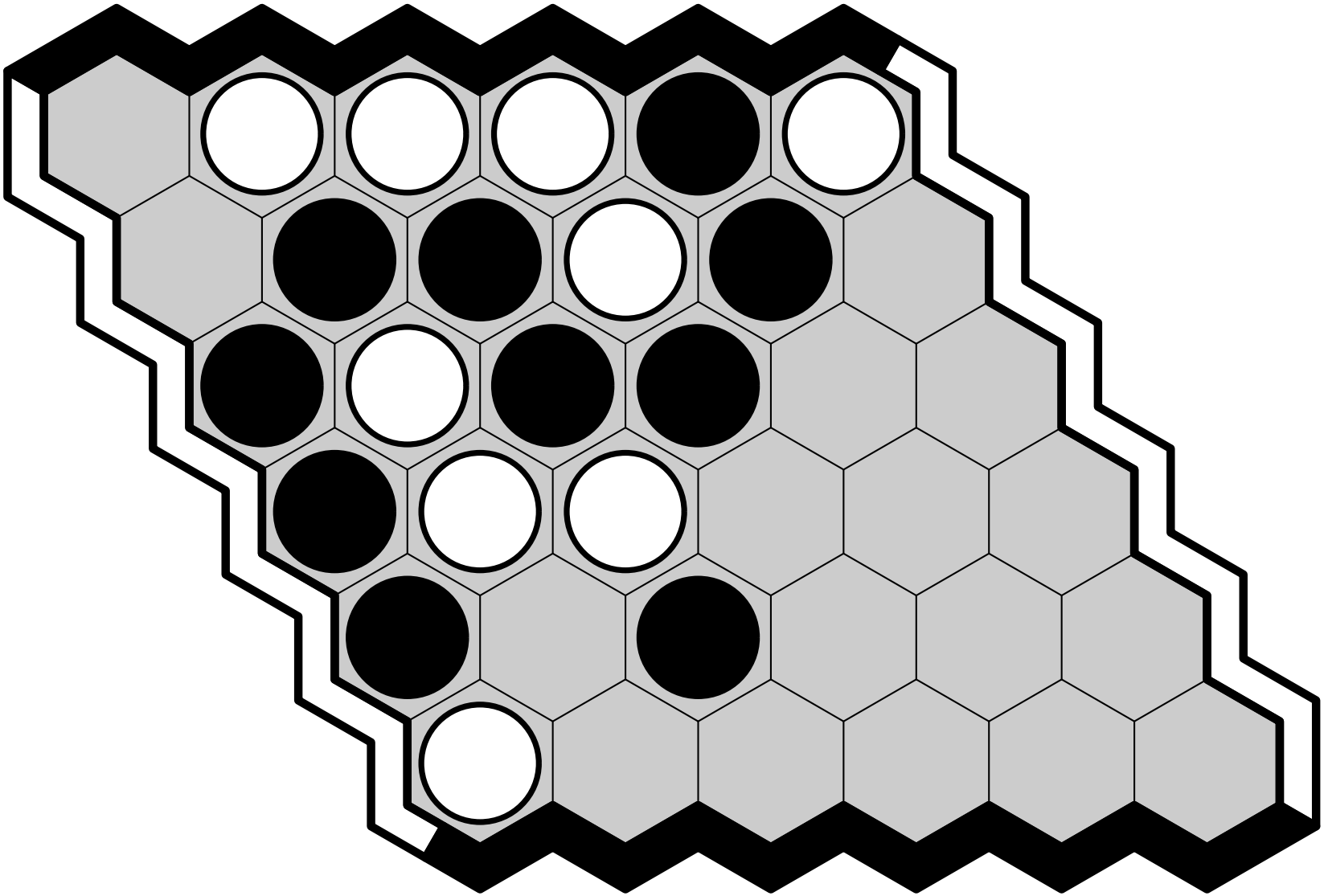
Gardner July '57 SciAm game of hex



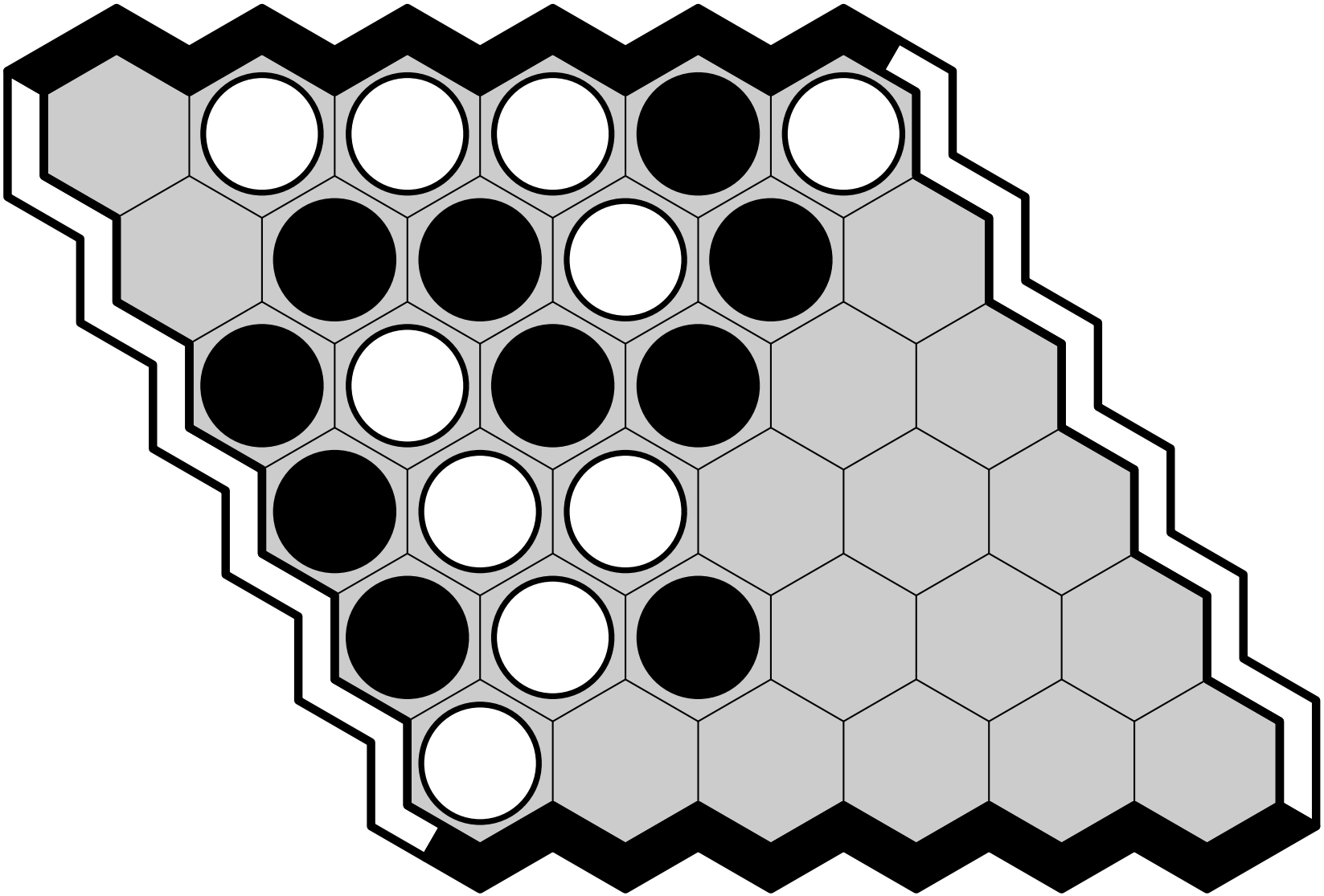
Gardner July '57 SciAm game of hex



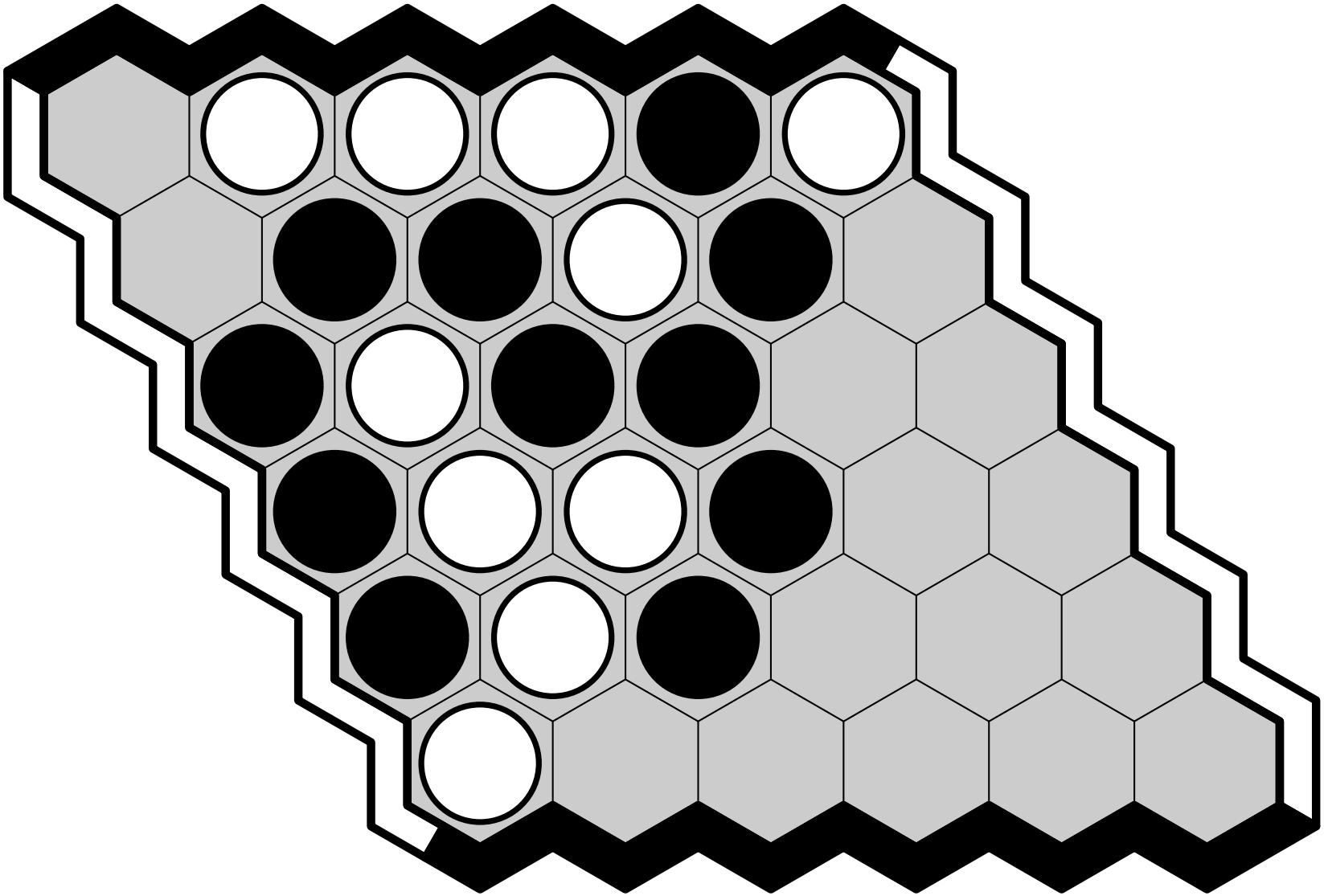
Gardner July '57 SciAm game of hex



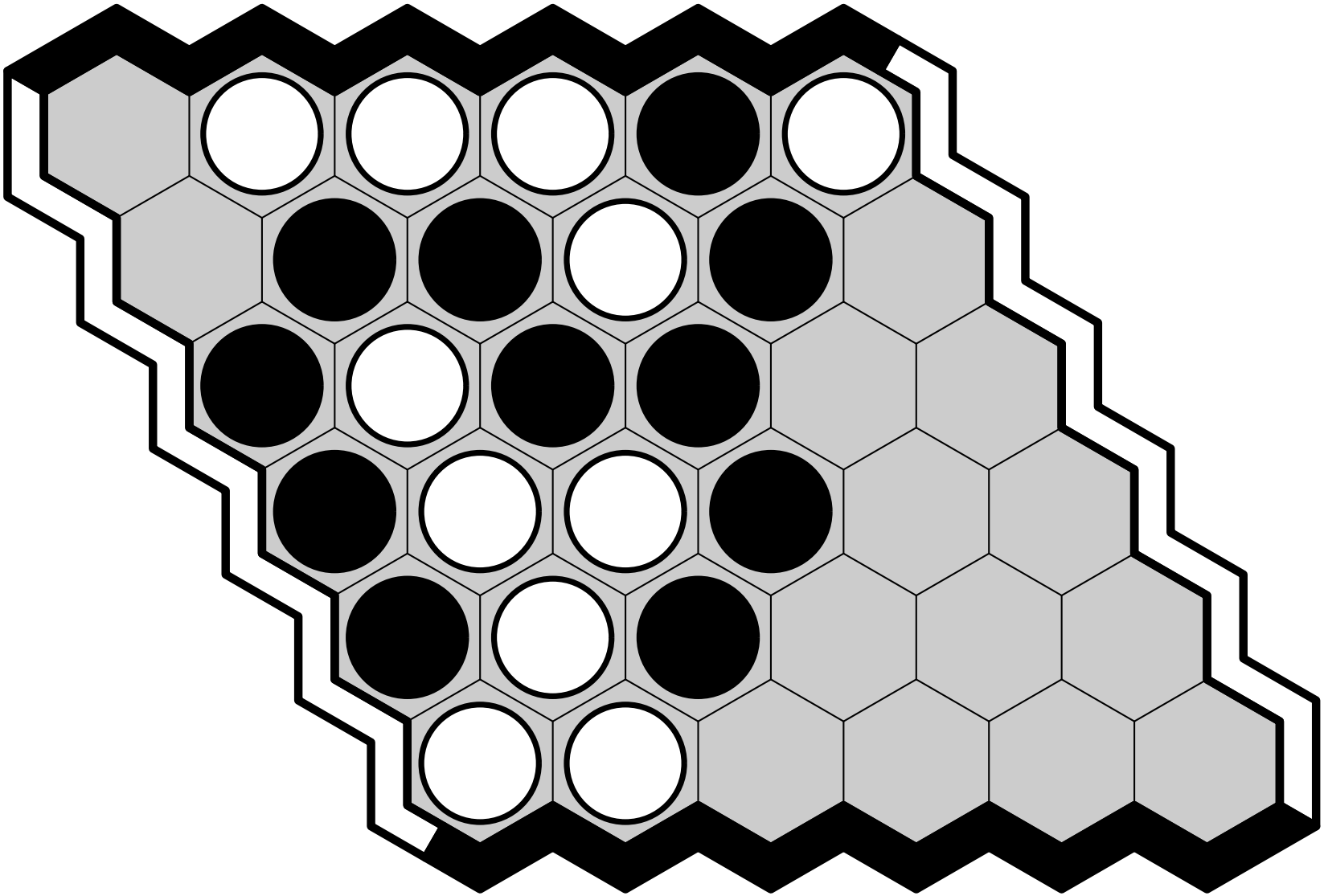
Gardner July '57 SciAm game of hex



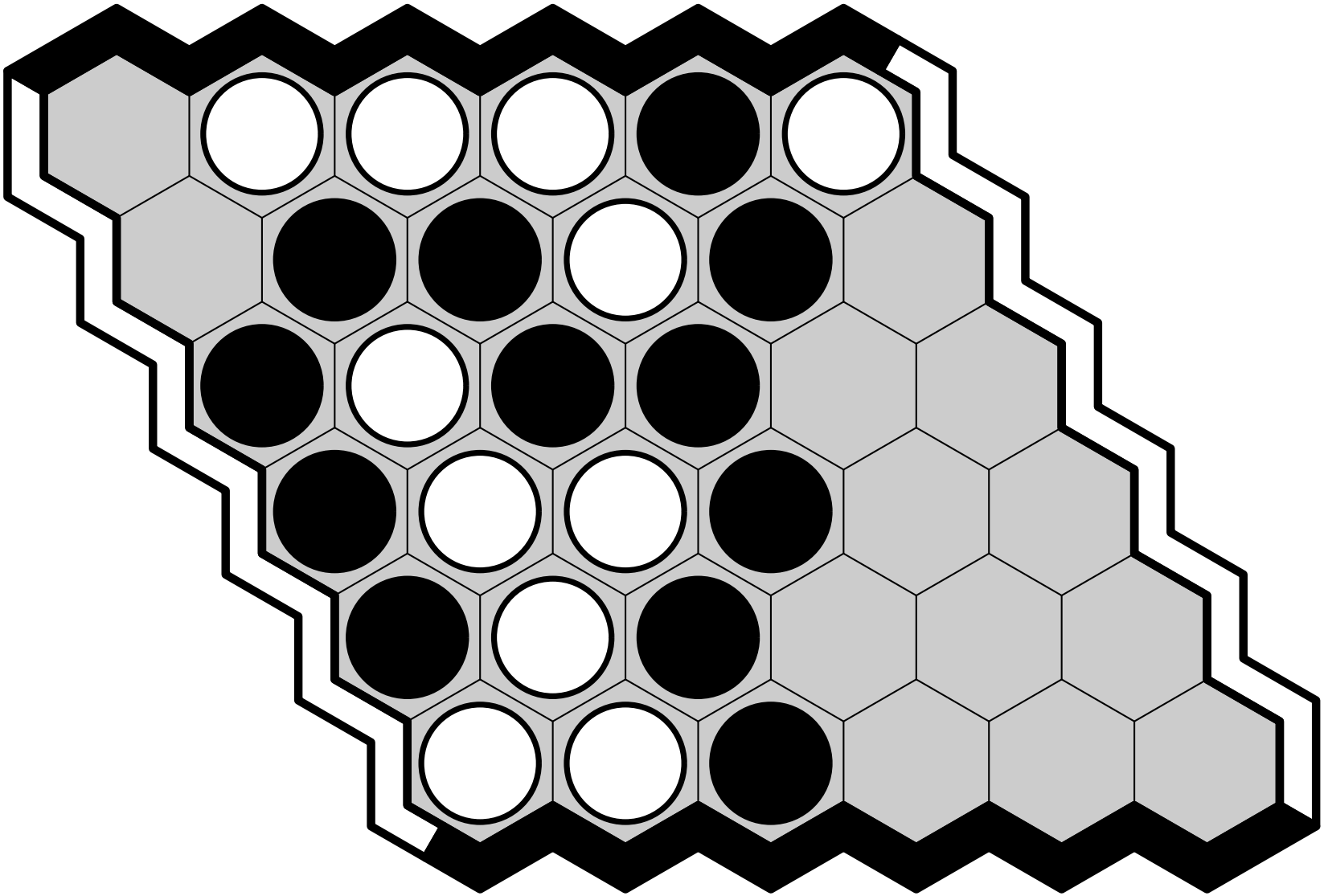
Gardner July '57 SciAm game of hex



Gardner July '57 SciAm game of hex



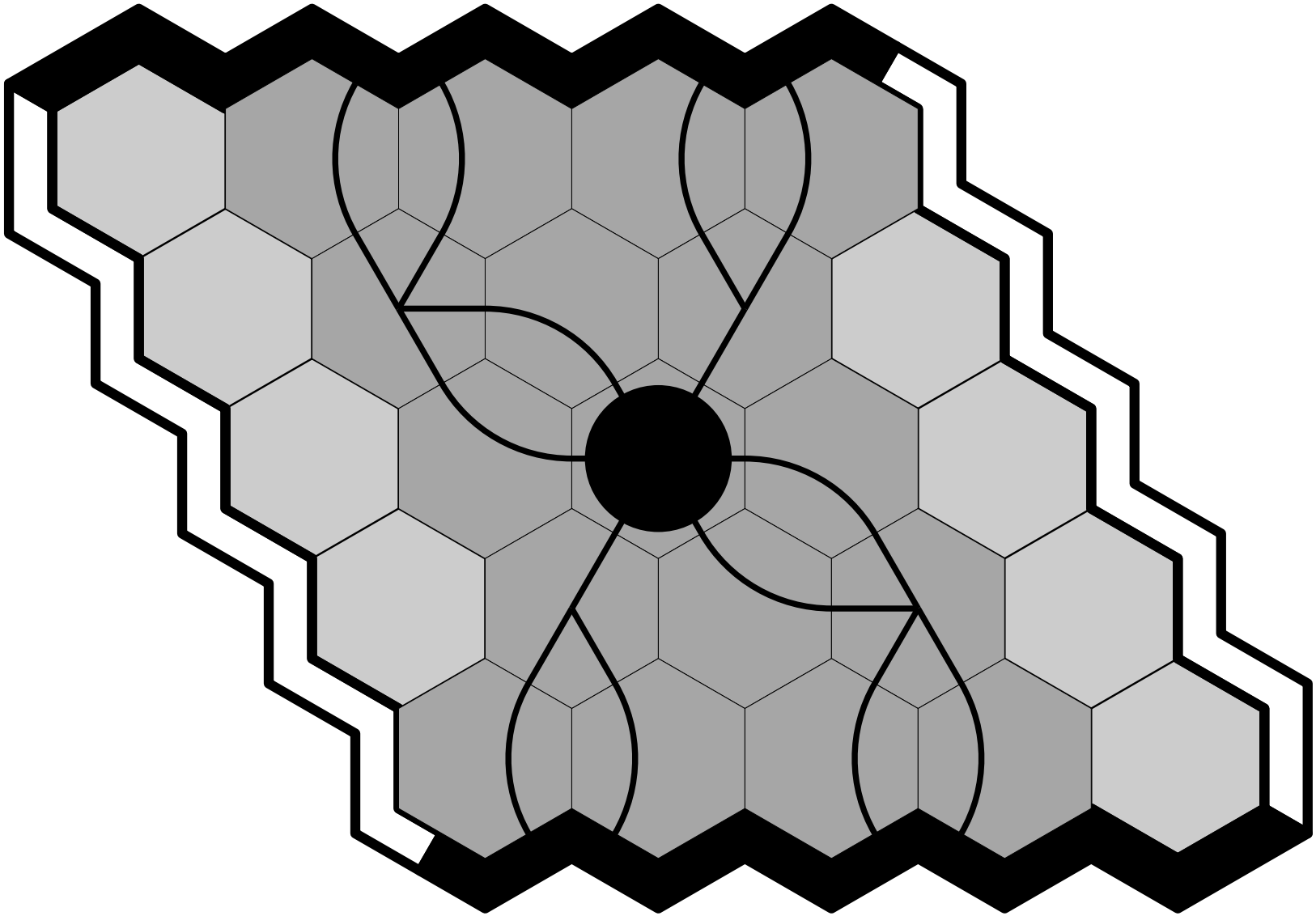
Gardner July '57 SciAm game of hex



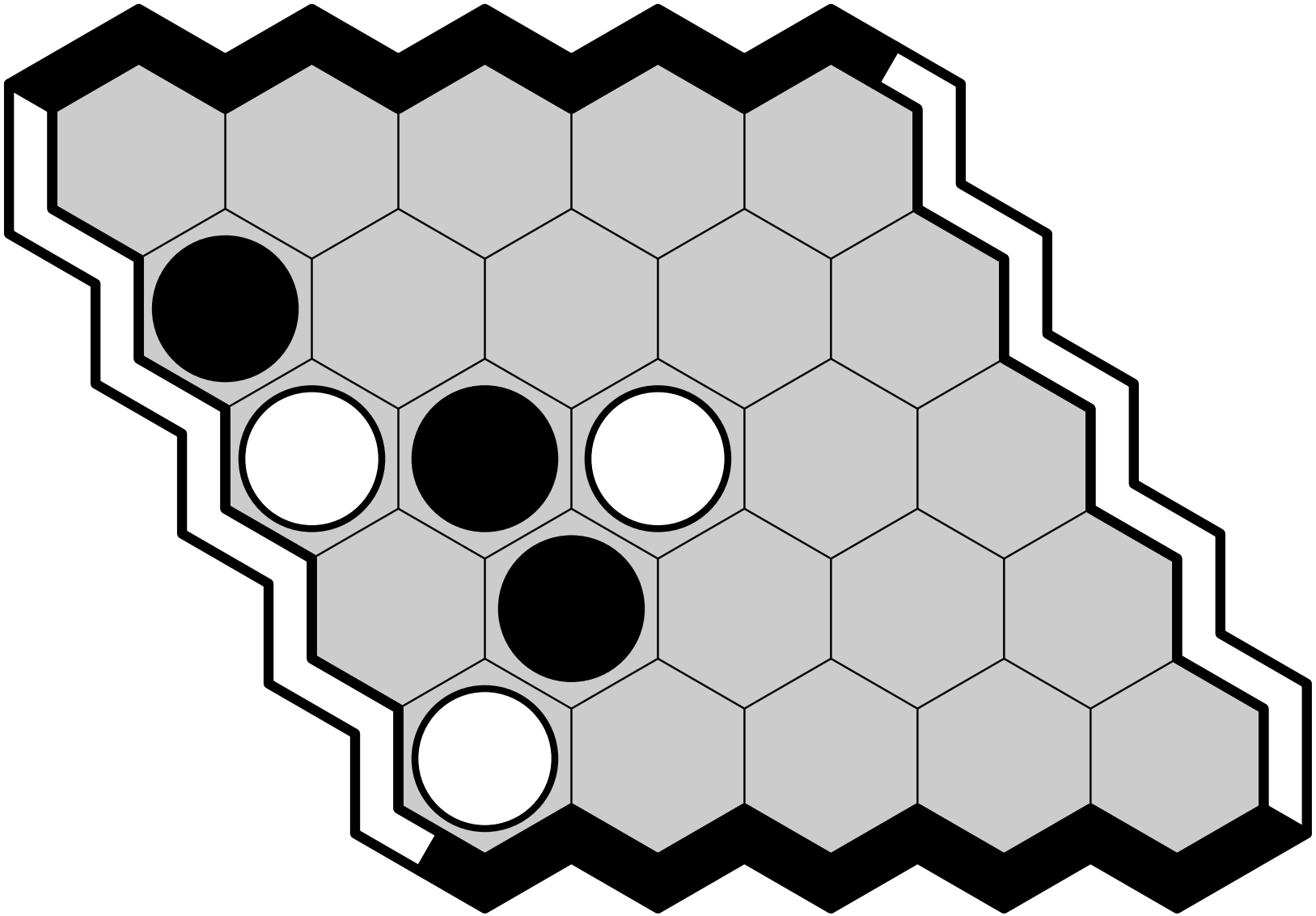
Gardner July '57 SciAm game of hex

- history
- draws are not possible
- nxn board: exists 1st-player-wins strategy
- nx(n−1) board: simple closer-sides-wins strategy
- variants, e.g. reverse hex

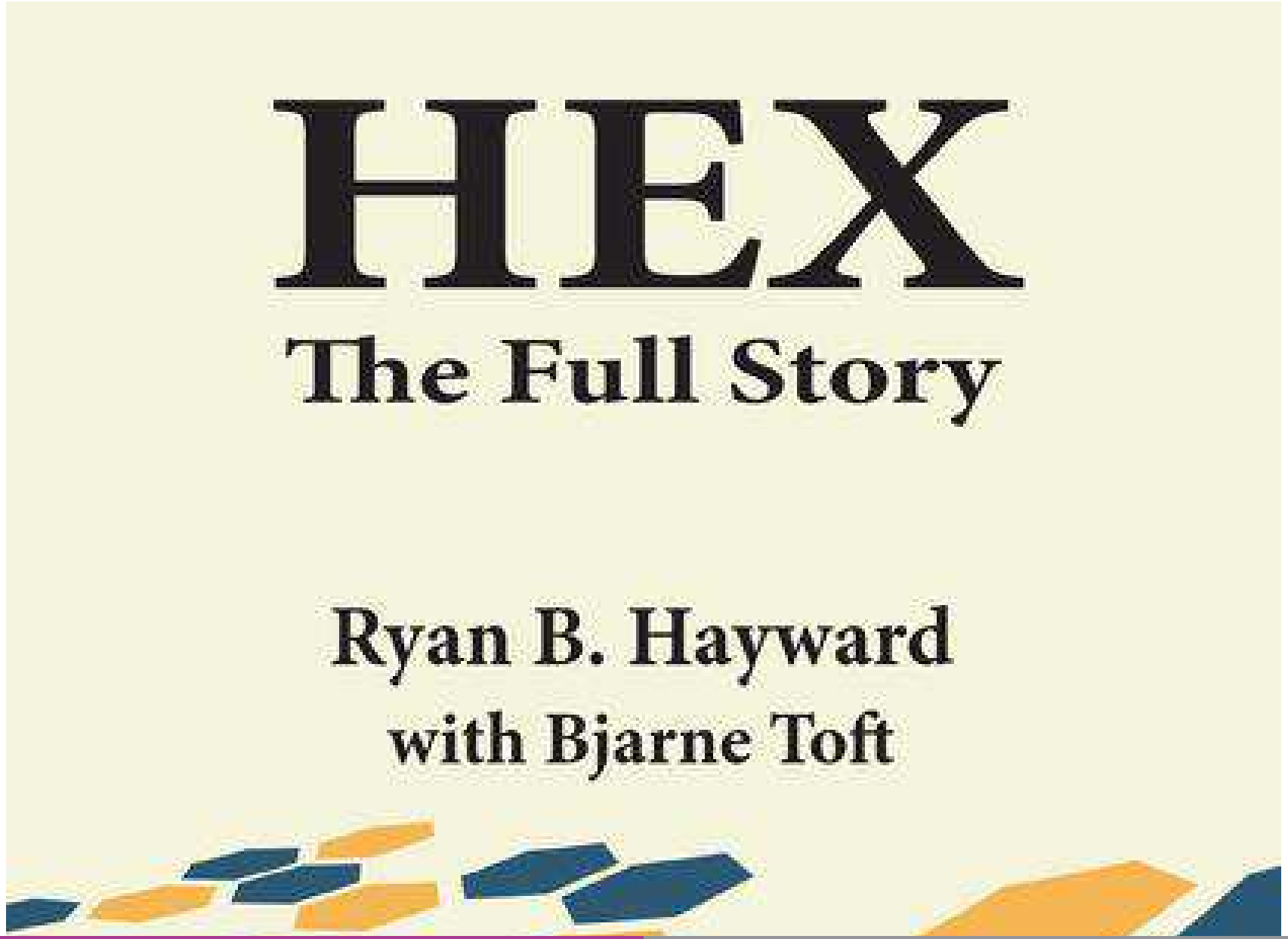
Gardner July '57 SciAm game of hex



Gardner July '57 SciAm game of hex

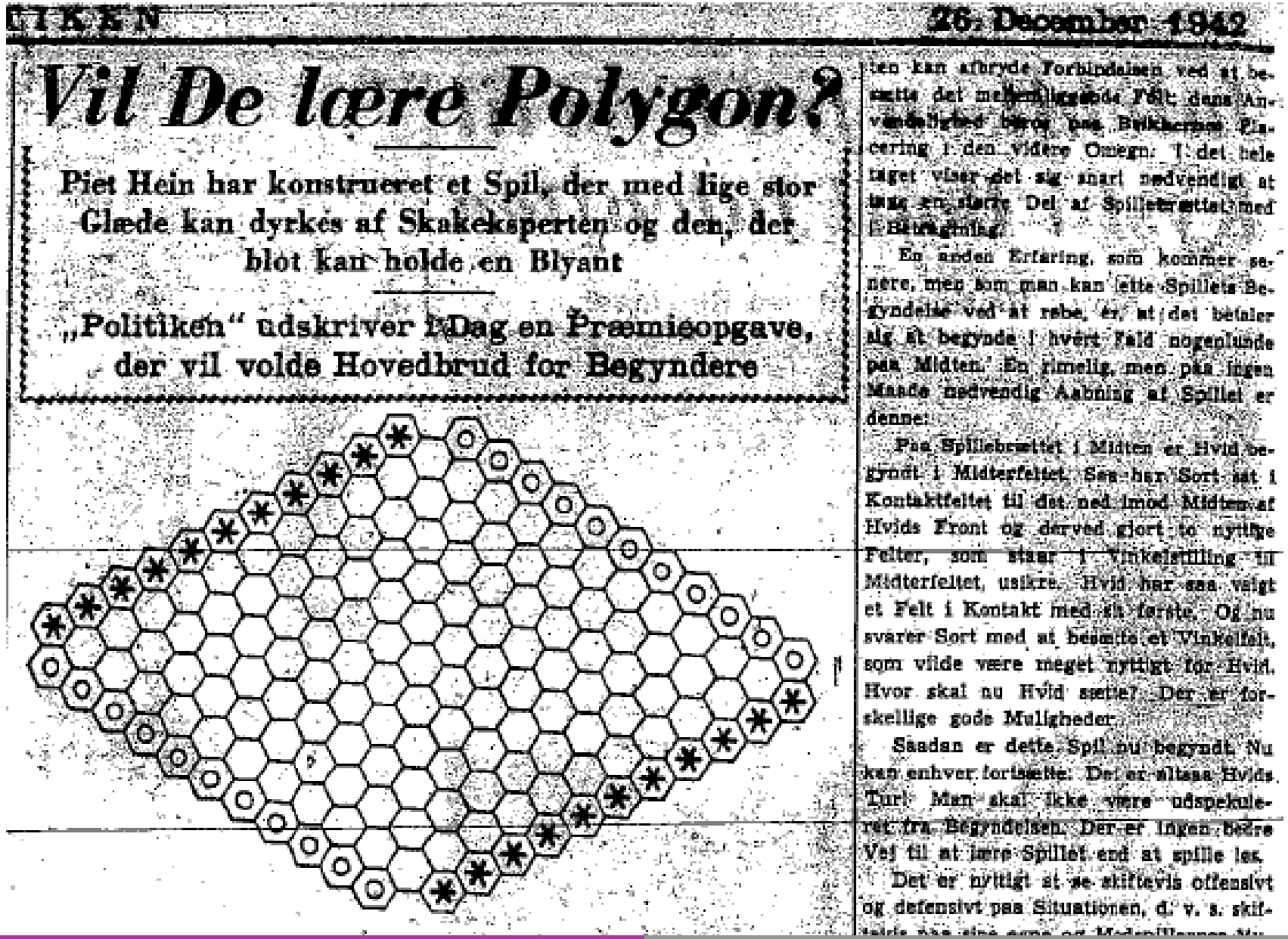


hex book one – history, 2019 CRC



hex book one – history, 2019 CRC

hex book one – history, 2019 CRC



TIKEN 26. December 1942

Vil De lære Polygon?

Piet Hein har konstrueret et Spil, der med lige stor Glæde kan dyrkes af Skakeksperten og den, der blot kan holde en Blyant

„Politiken" udskriver i Dag en Præmieopgave, der vil volde Hovedbrud for Begyndere

ten kan afbryde Forbindelsen ved at besætte det mellemliggende Felt; dens Anvendelighed bero[r] paa Brikkernes Placering i den videre Omegn. I det hele taget viser det sig snart nødvendigt at tage en større Del af Spillepladen med i Betragtning.

En anden Erfaring, som kommer senere, men som man kan lette Spillets Begyndelse ved at rebe, er, at det betaler sig at begynde i hvert Fald nogenlunde paa Midten. En rimelig, men paa ingen Maade nødvendig Aabning af Spillet er denne:

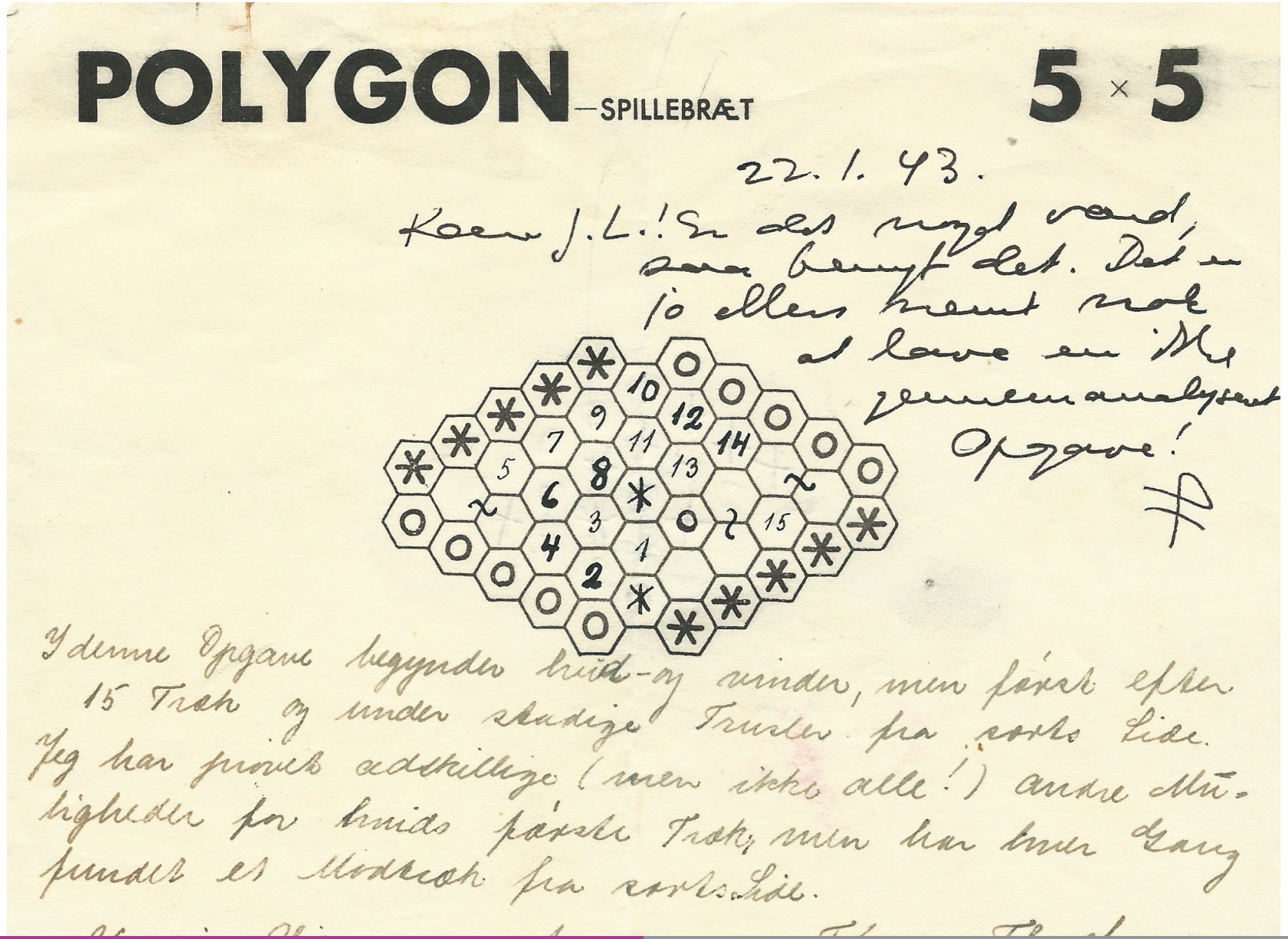
Paa Spillebrættet i Midten er Hvid begyndt i Midterfeltet. Sea har Sort sat i Kontaktfeltet til det ned imod Midten af Hvids Front og derved gjort to nyttige Felter, som staar i Vinkelstilling til Midterfeltet, usikre. Hvid har saa valgt et Felt i Kontakt med sit første. Og nu svarer Sort med at besætte et Vinkelfelt, som vilde være meget nyttigt for Hvid. Hvor skal nu Hvid sætte? Der er forskellige gode Muligheder.

Saadan er dette Spil nu begyndt. Nu kan enhver fortsætte. Det er altsaa Hvids Tur. Man skal ikke være udspekuleret fra Begyndelsen. Der er ingen bedre Vej til at lære Spillet end at spille løs.

Det er nyttigt at se skiftevis offensivt og defensivt paa Situationen, d. v. s. skiftevis paa sine egne og Modspillerens Mu-

hex book one – history, 2019 CRC

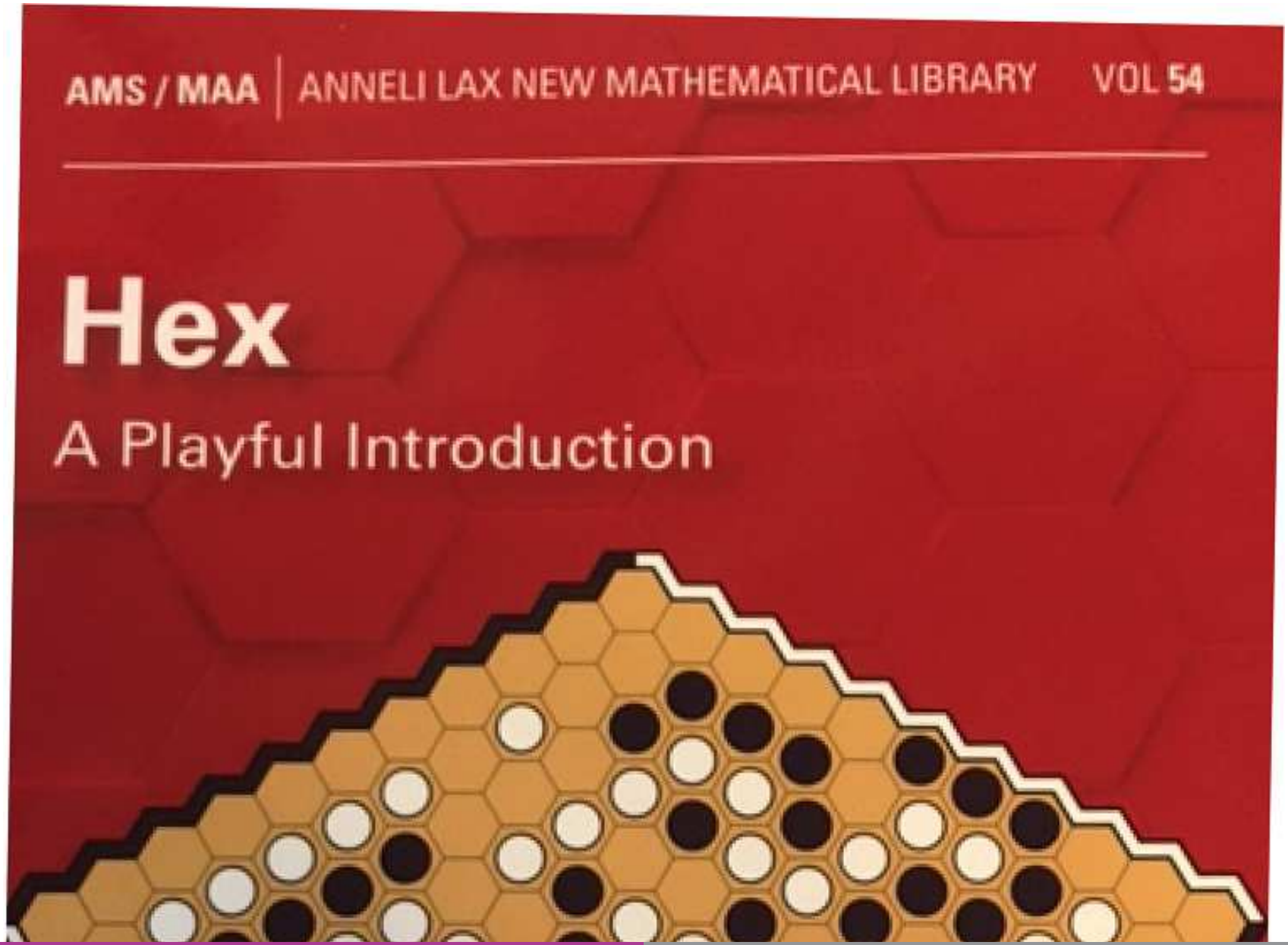
hex book one – history, 2019 CRC



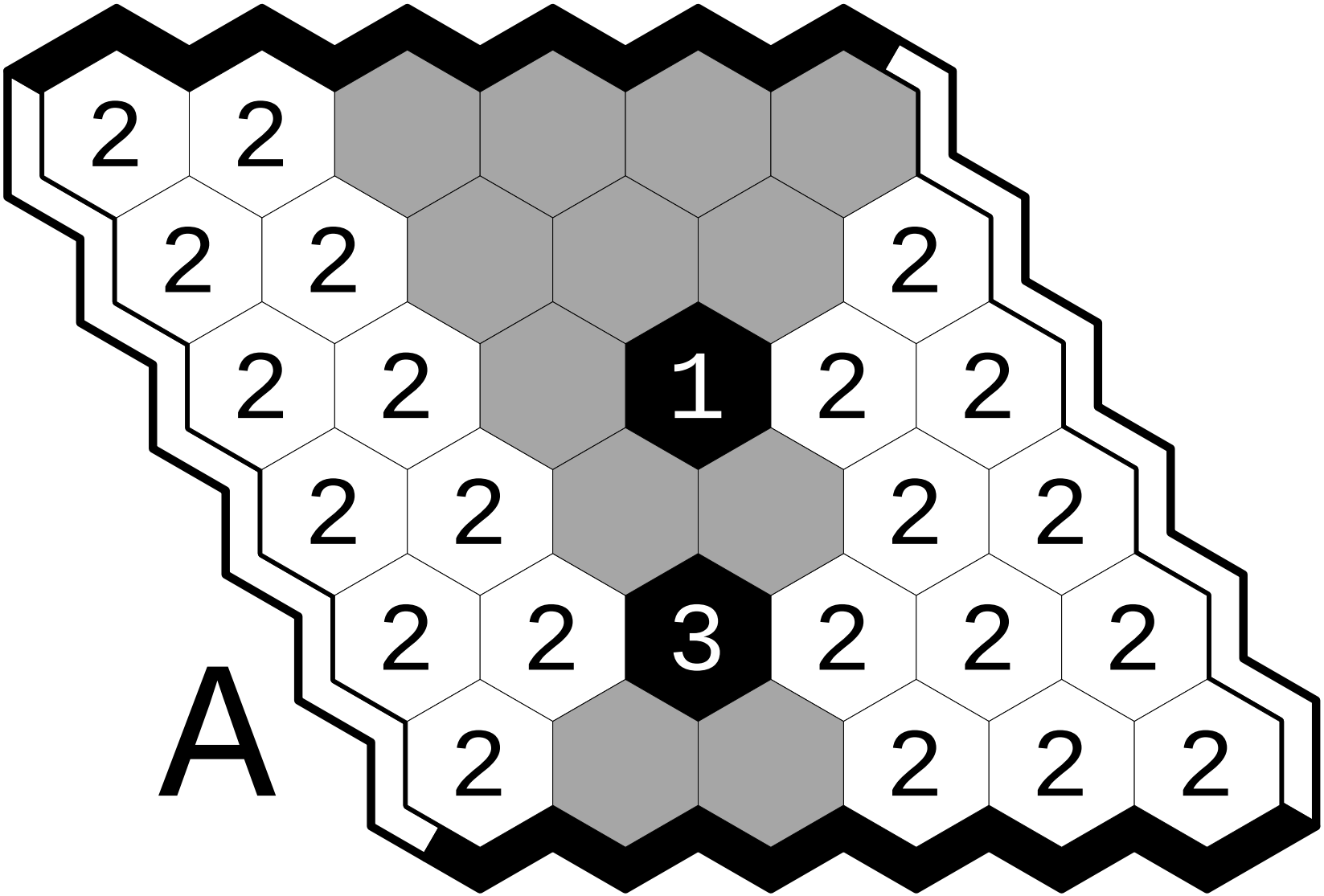
hex book one – history, 2019 CRC

https://www.routledge.com/hex-Inside-and-Out-

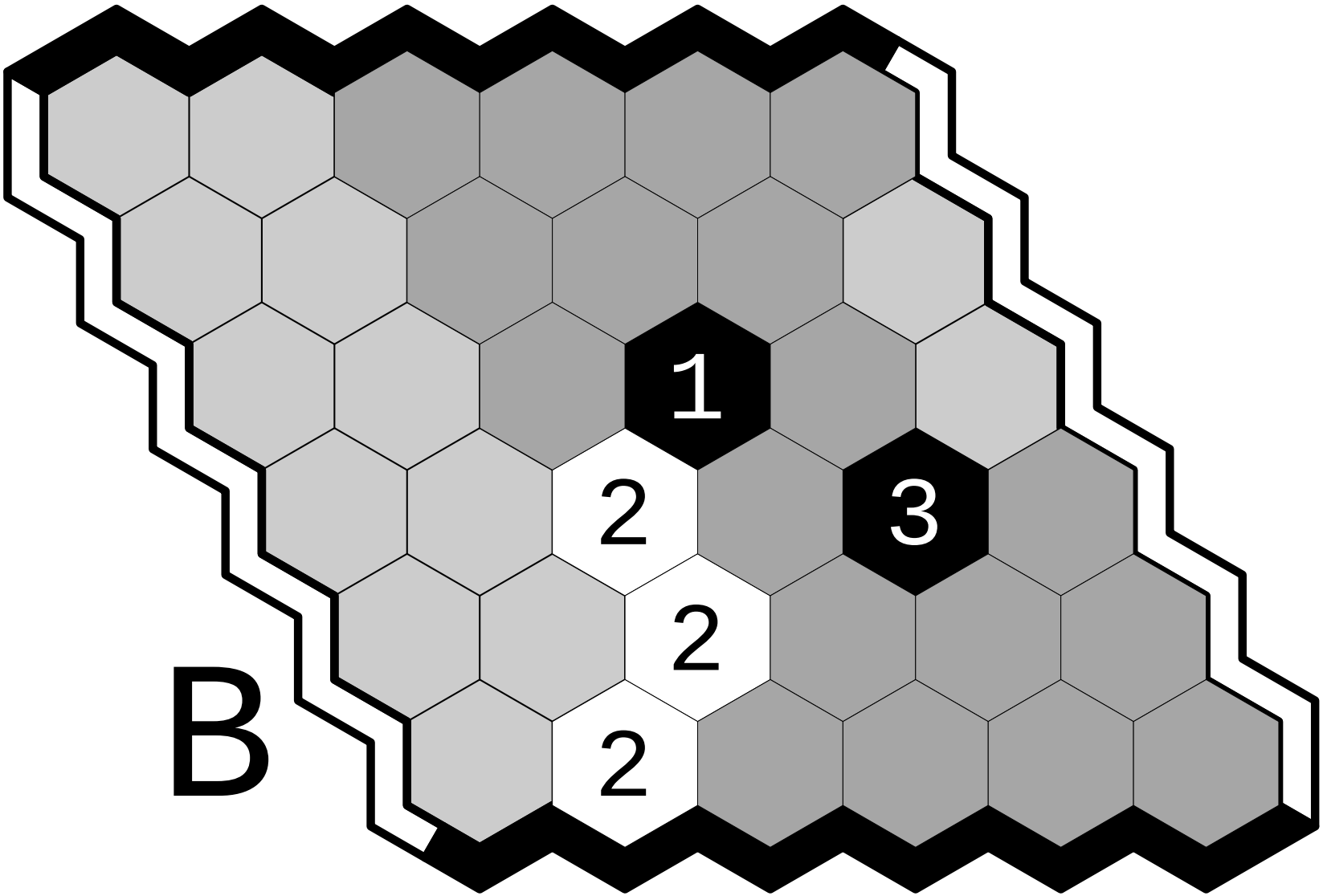
hex book two – intro, 2022 MAA



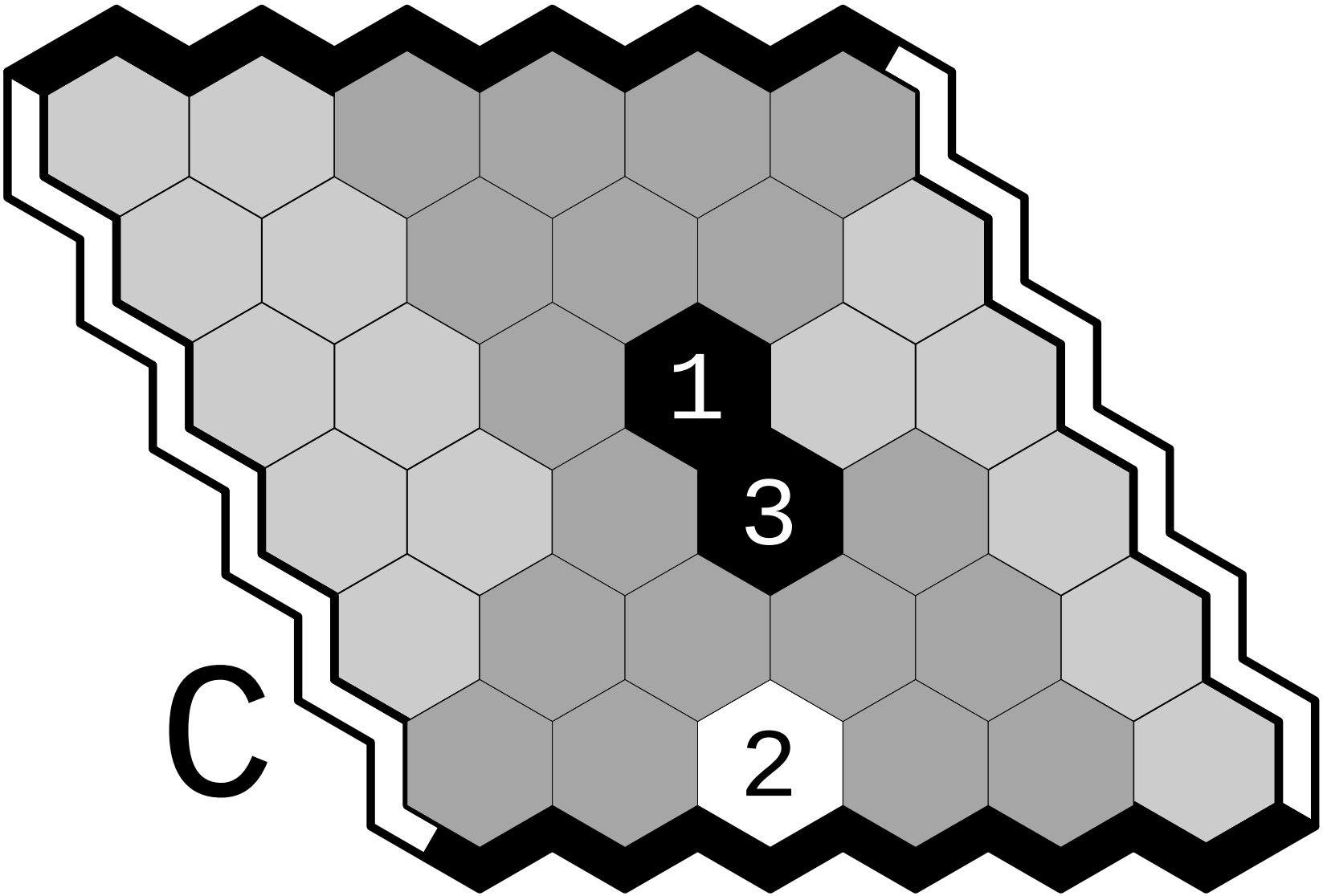
hex book two – intro, 2022 MAA



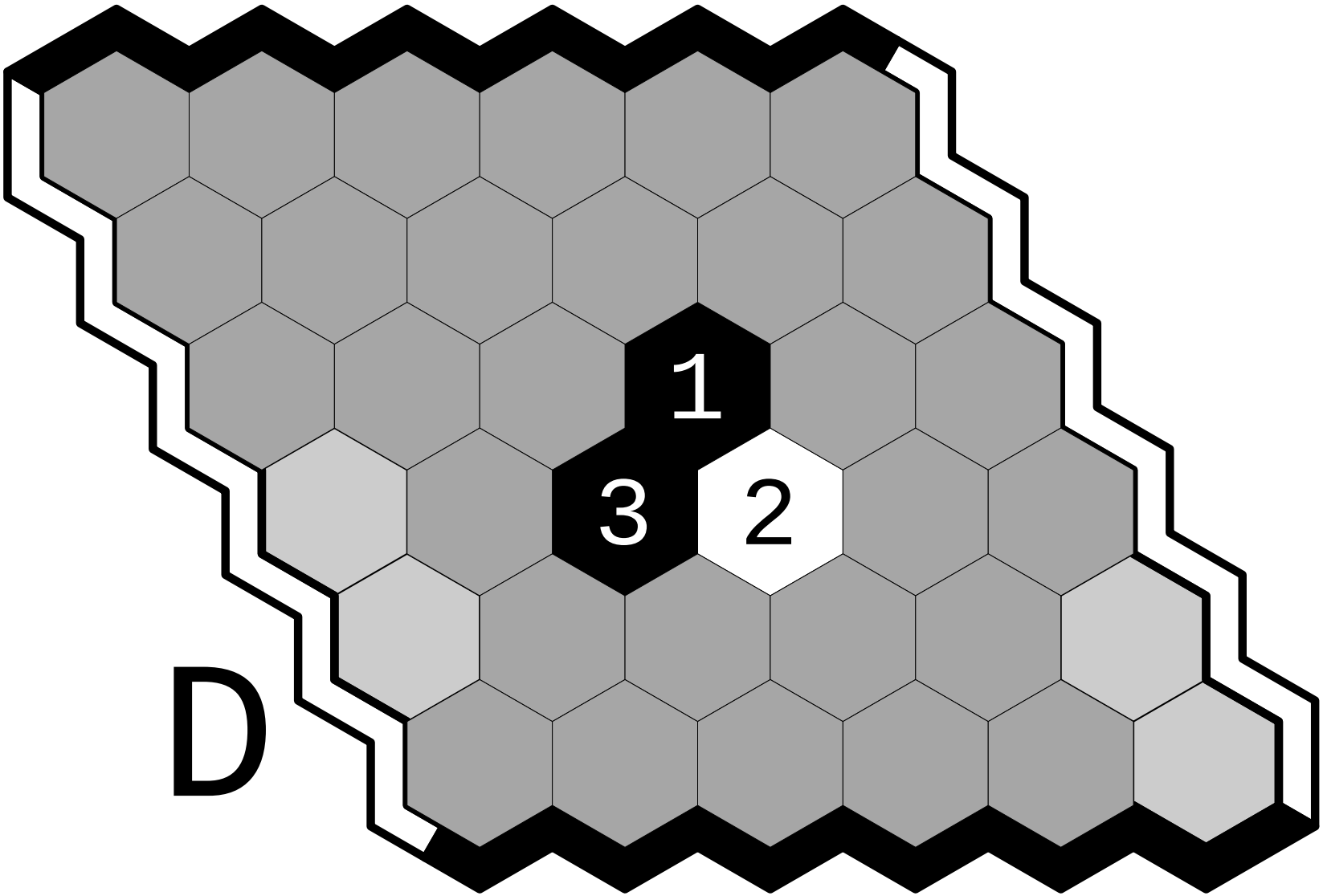
hex book two – intro, 2022 MAA



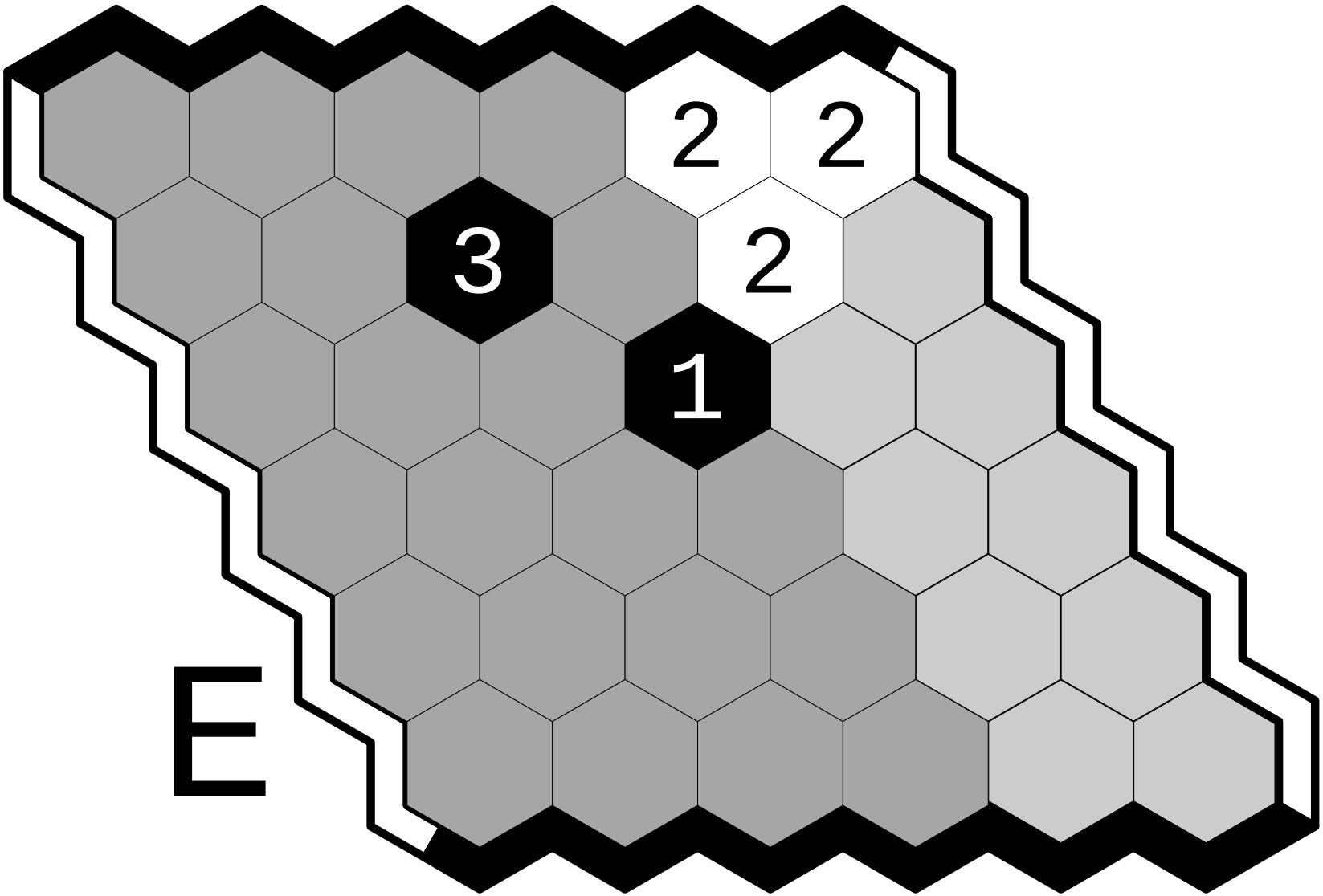
hex book two – intro, 2022 MAA



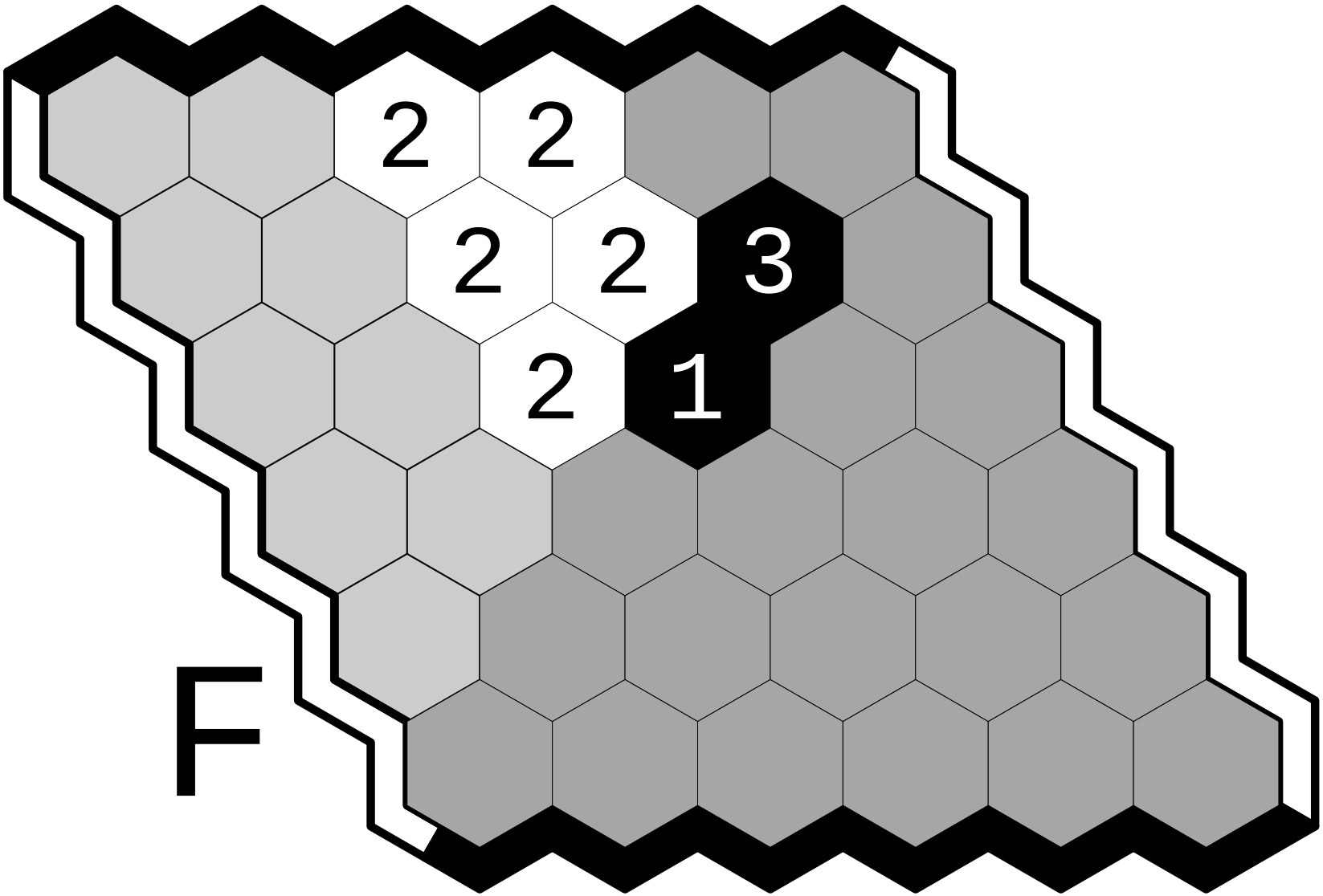
hex book two – intro, 2022 MAA



hex book two – intro, 2022 MAA



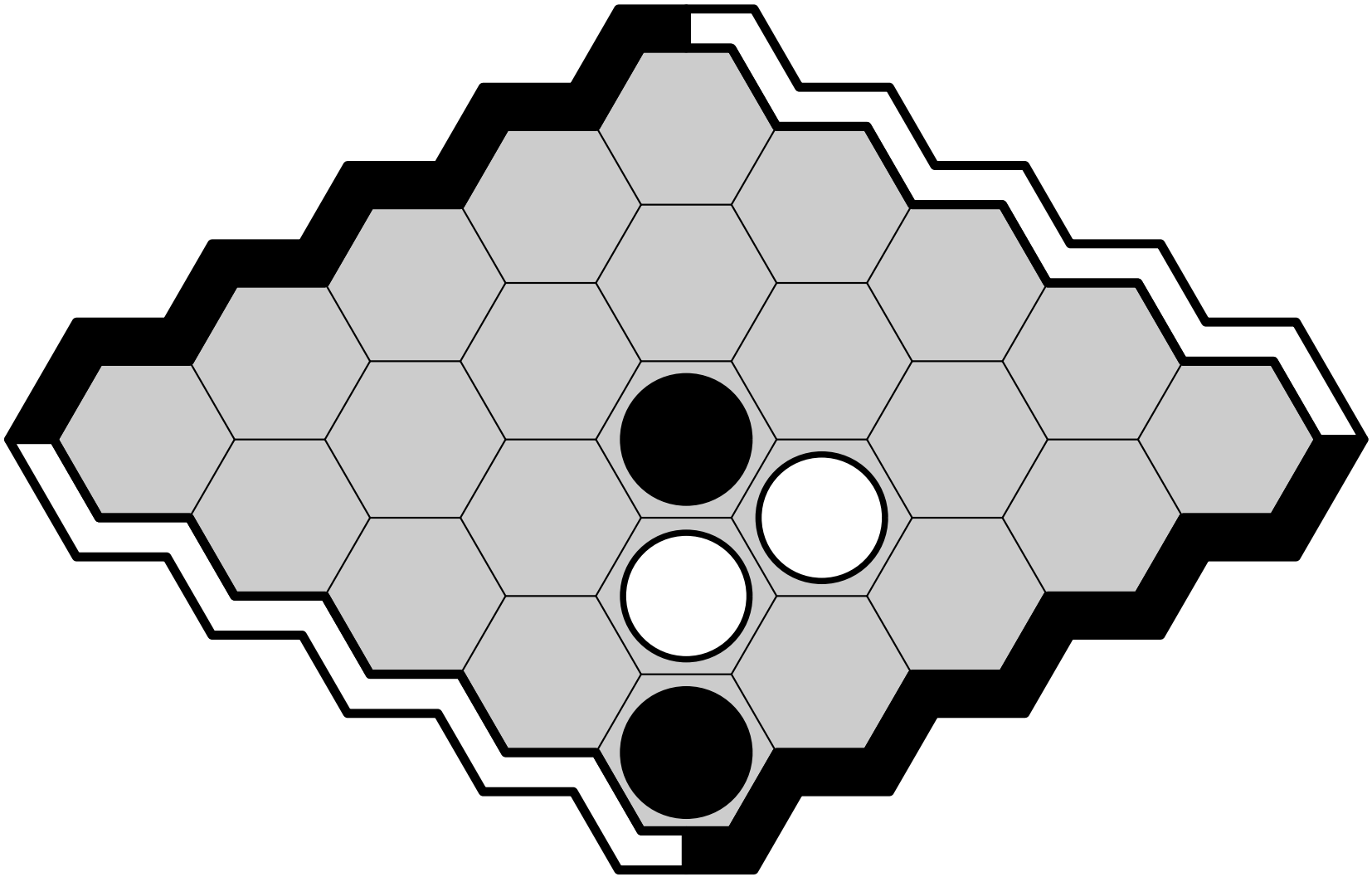
hex book two – intro, 2022 MAA



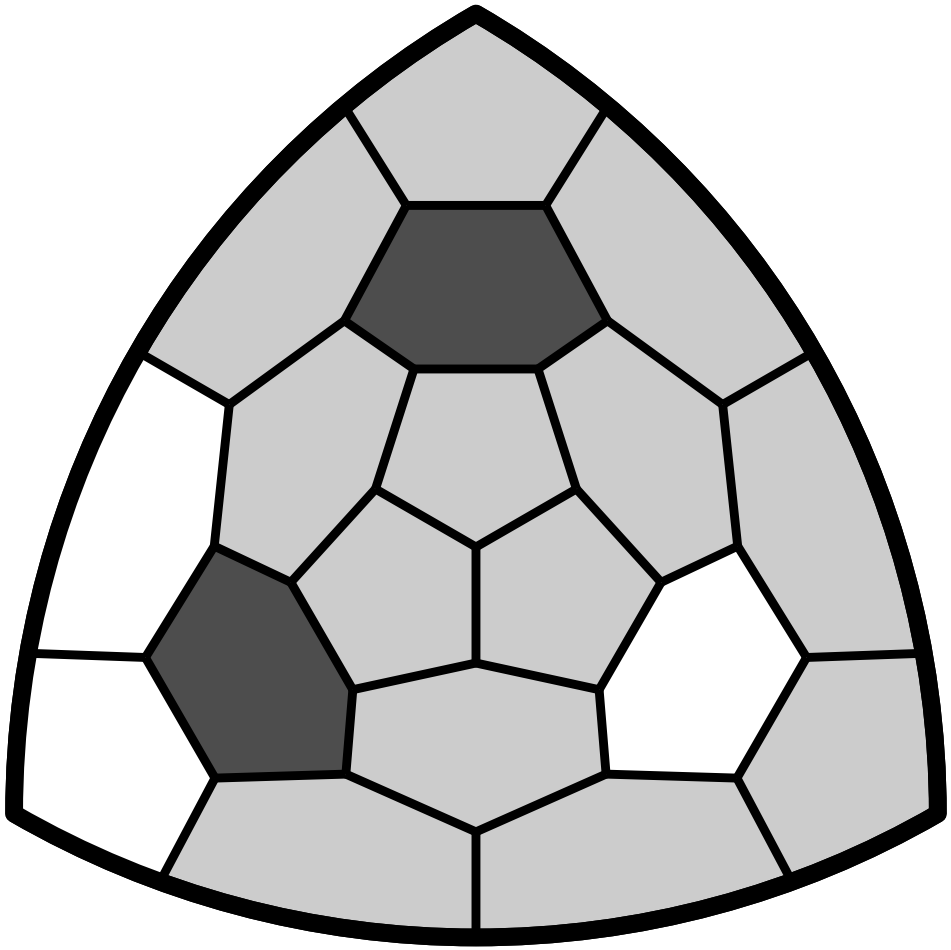
hex book two – intro, 2022 MAA

https://bookstore.ams.org/nml-54/

hex puzzle (based on Karen T) black to play



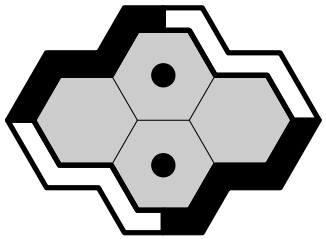
bonus puzzle – Y game – black to play



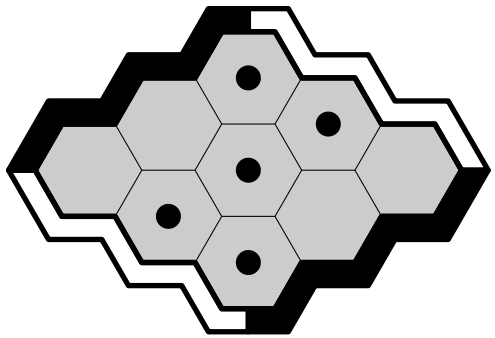
to win, join all three sides

WINNING 1ST-MOVES

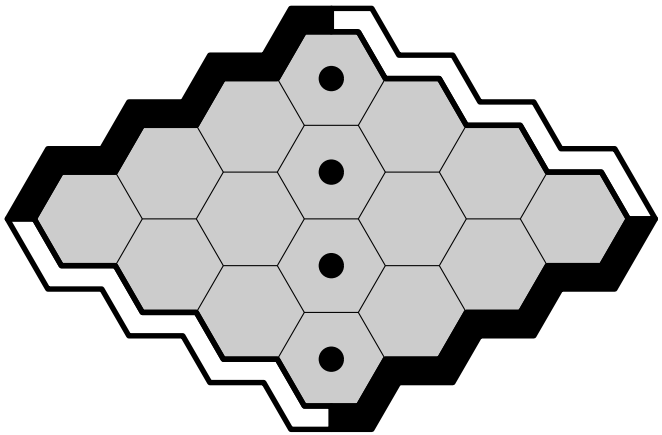
WINNING 1ST-MOVES



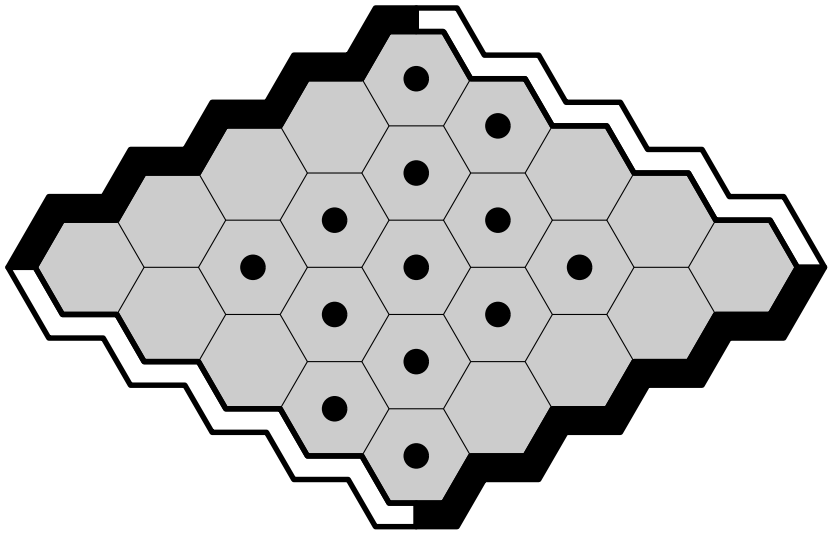
WINNING 1ST-MOVES



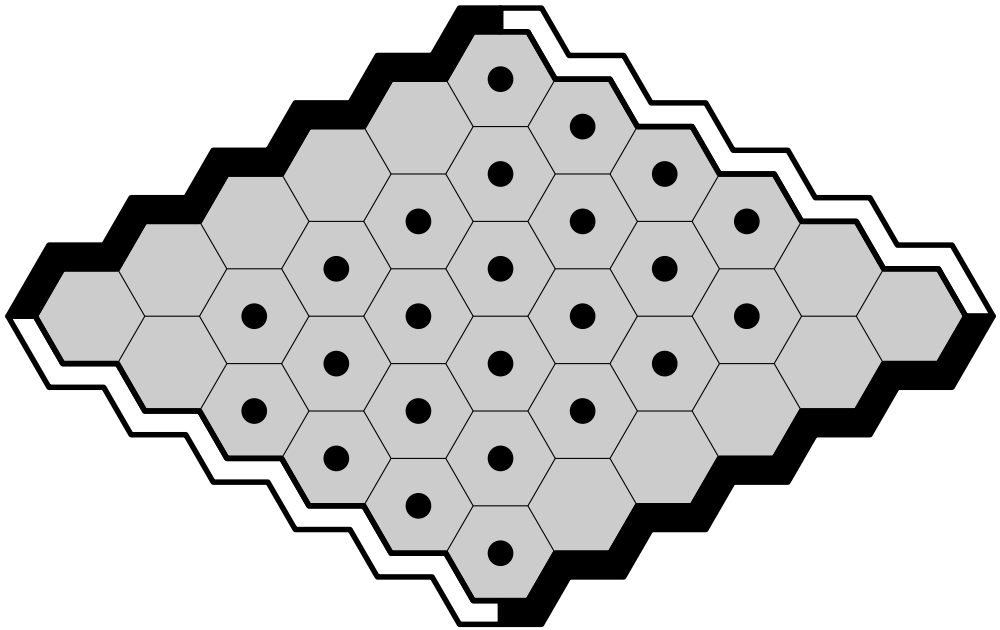
WINNING 1ST-MOVES



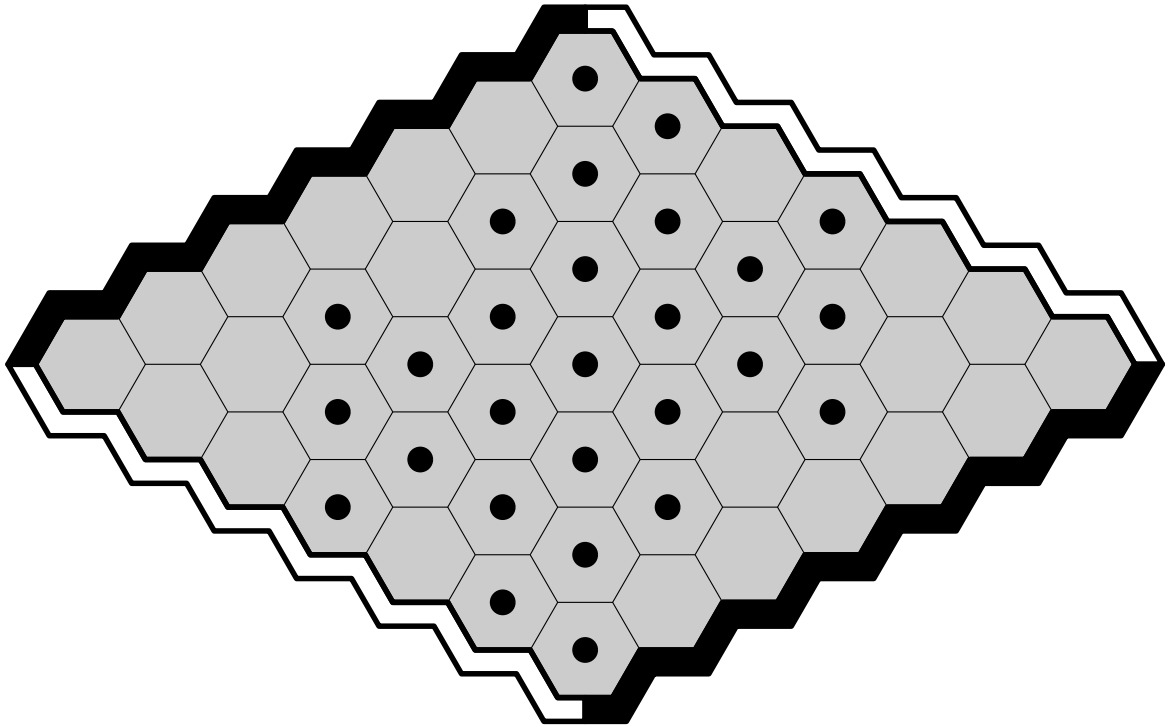
WINNING 1ST-MOVES



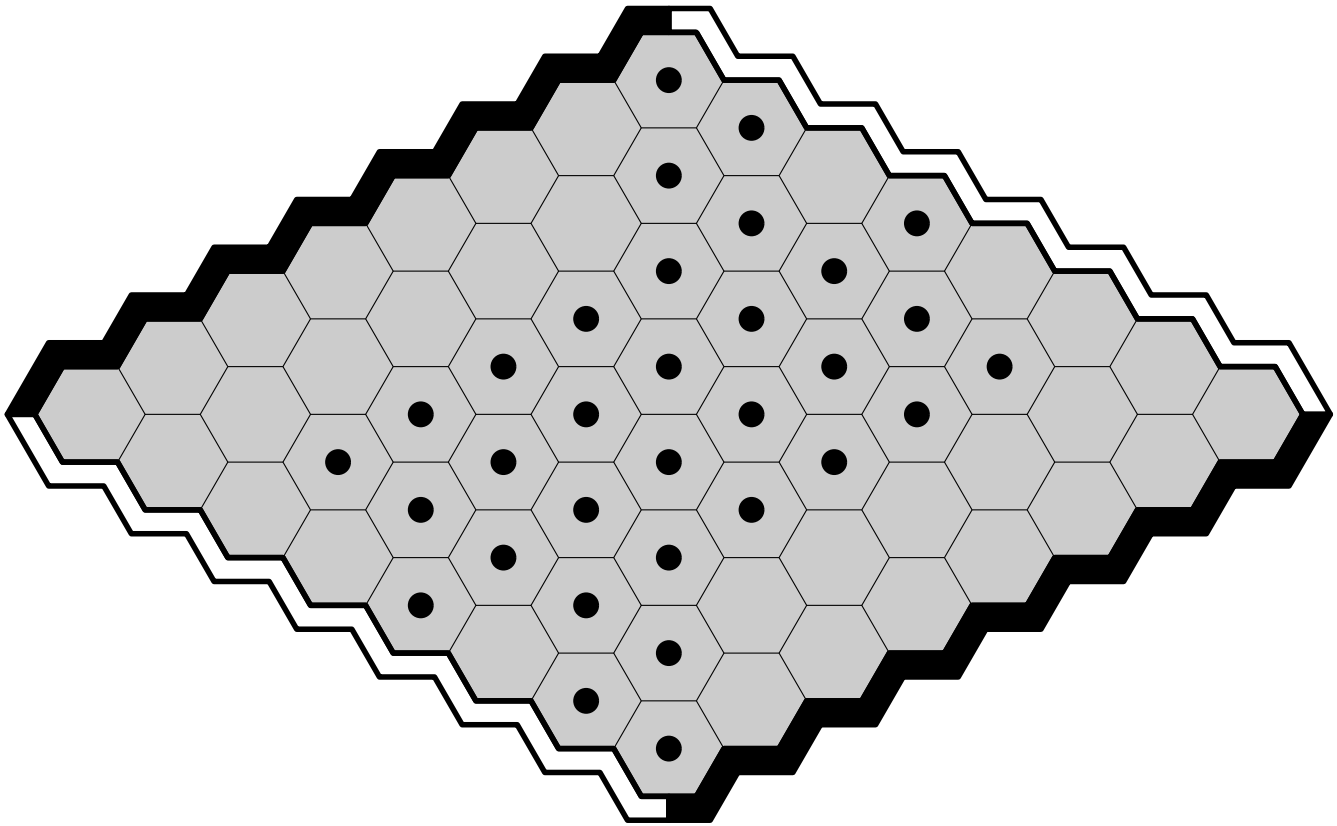
winning 1st-moves 1995 Enderton



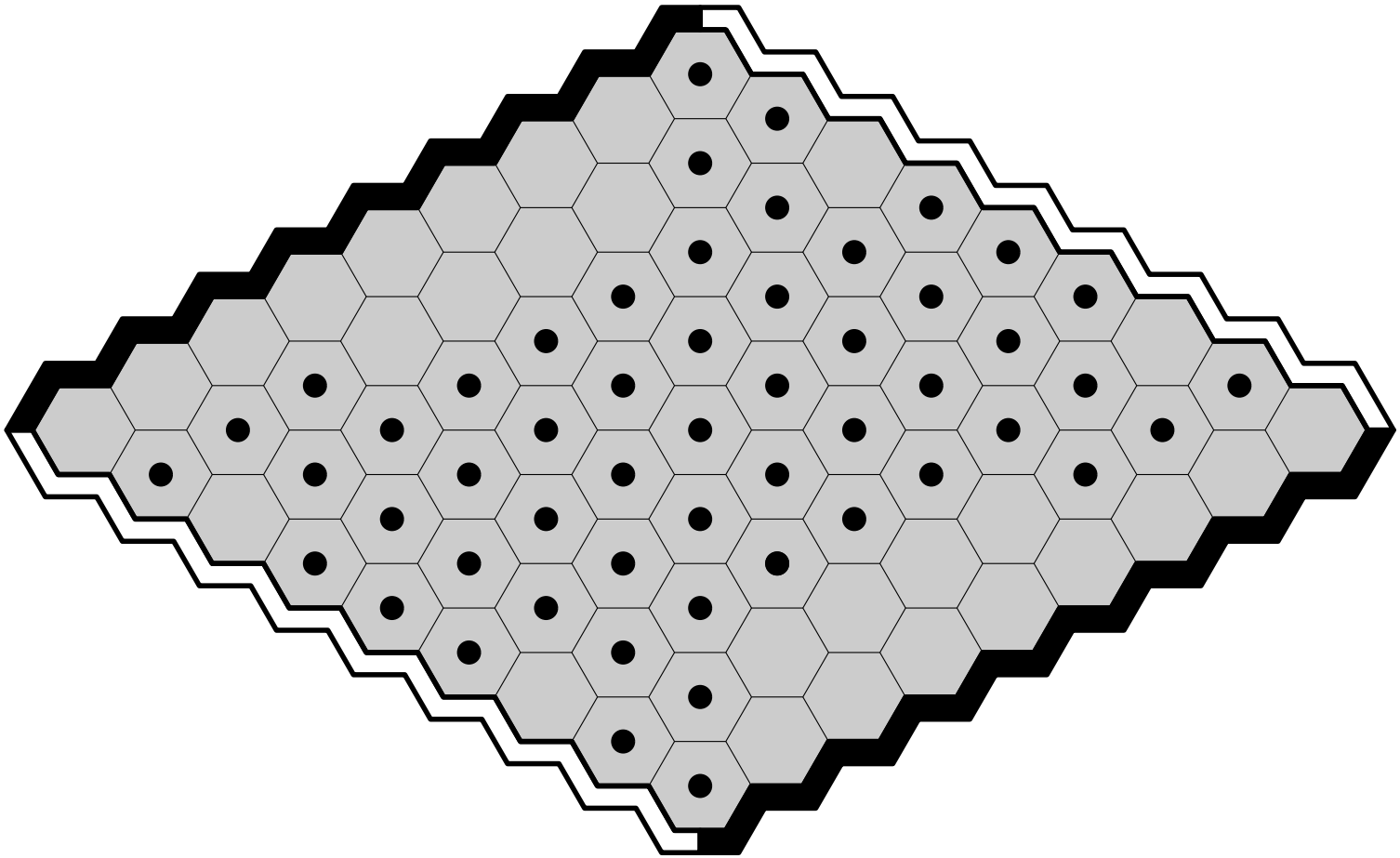
WINNING 1ST-MOVES 2004 HBJPvR



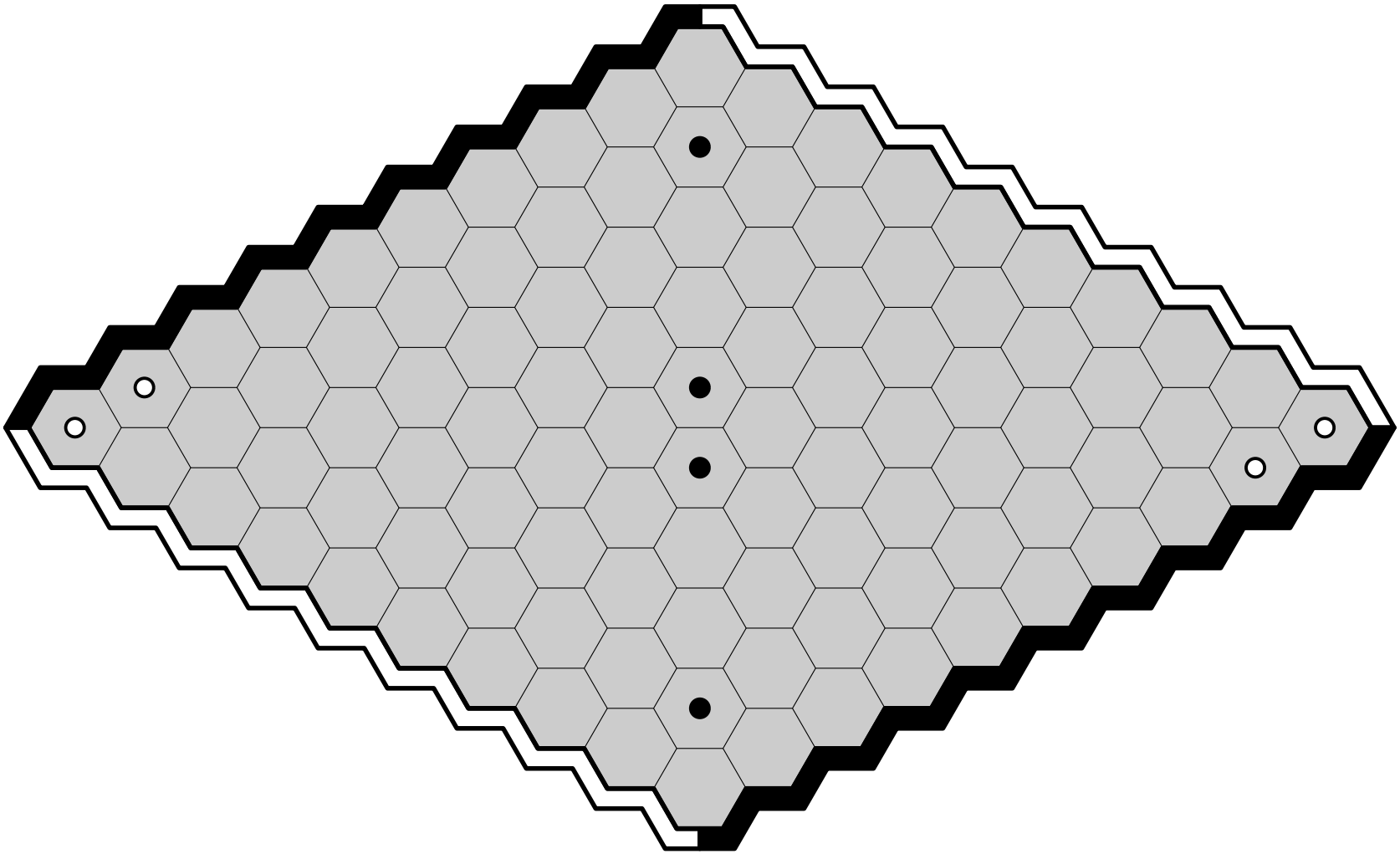
WINNING 1ST-MOVES 2009 HAH



WINNING 1ST-MOVES 2013 AHHP



WINNING 1ST-MOVES 2014 PH



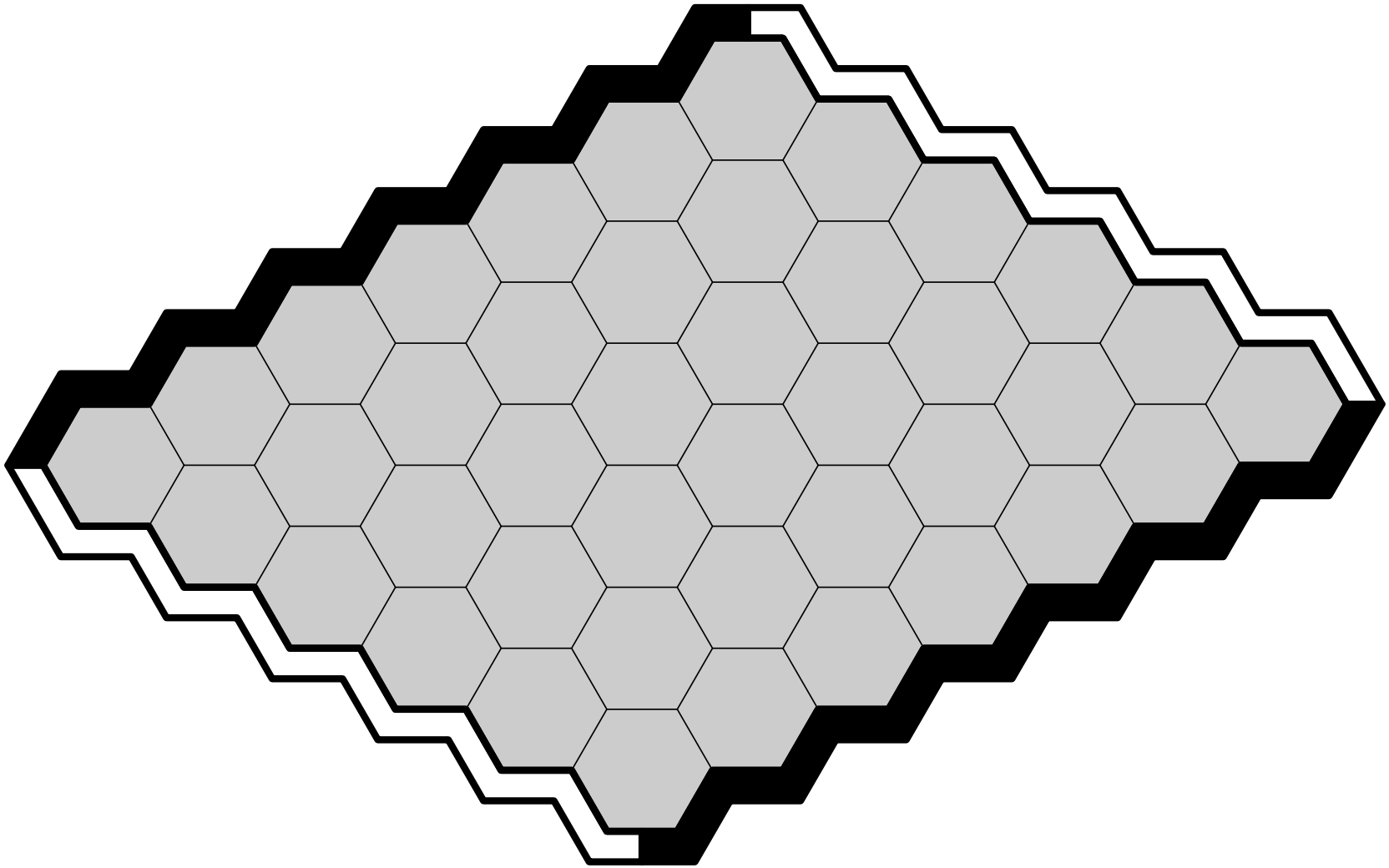
HEX THEOREMS

- no draws
- $n \times n$ hex, $n \geq 1$ 1st-player

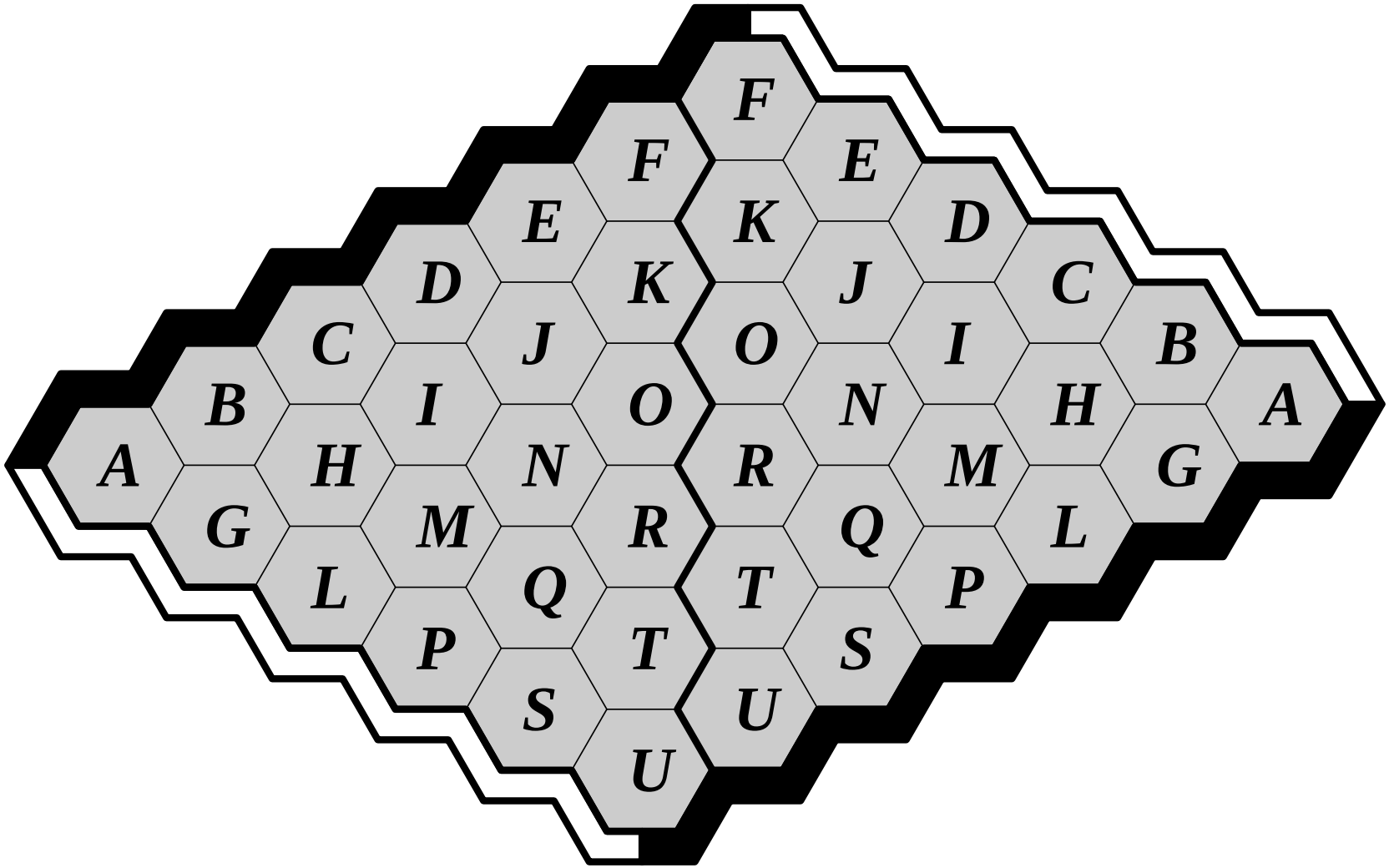
  exists winning strategy
- $n \times m$ hex, $1 \leq n < m$ joins-closer-edges player

  explicit winning strategy
- who wins arbitrary positions ? P-space complete

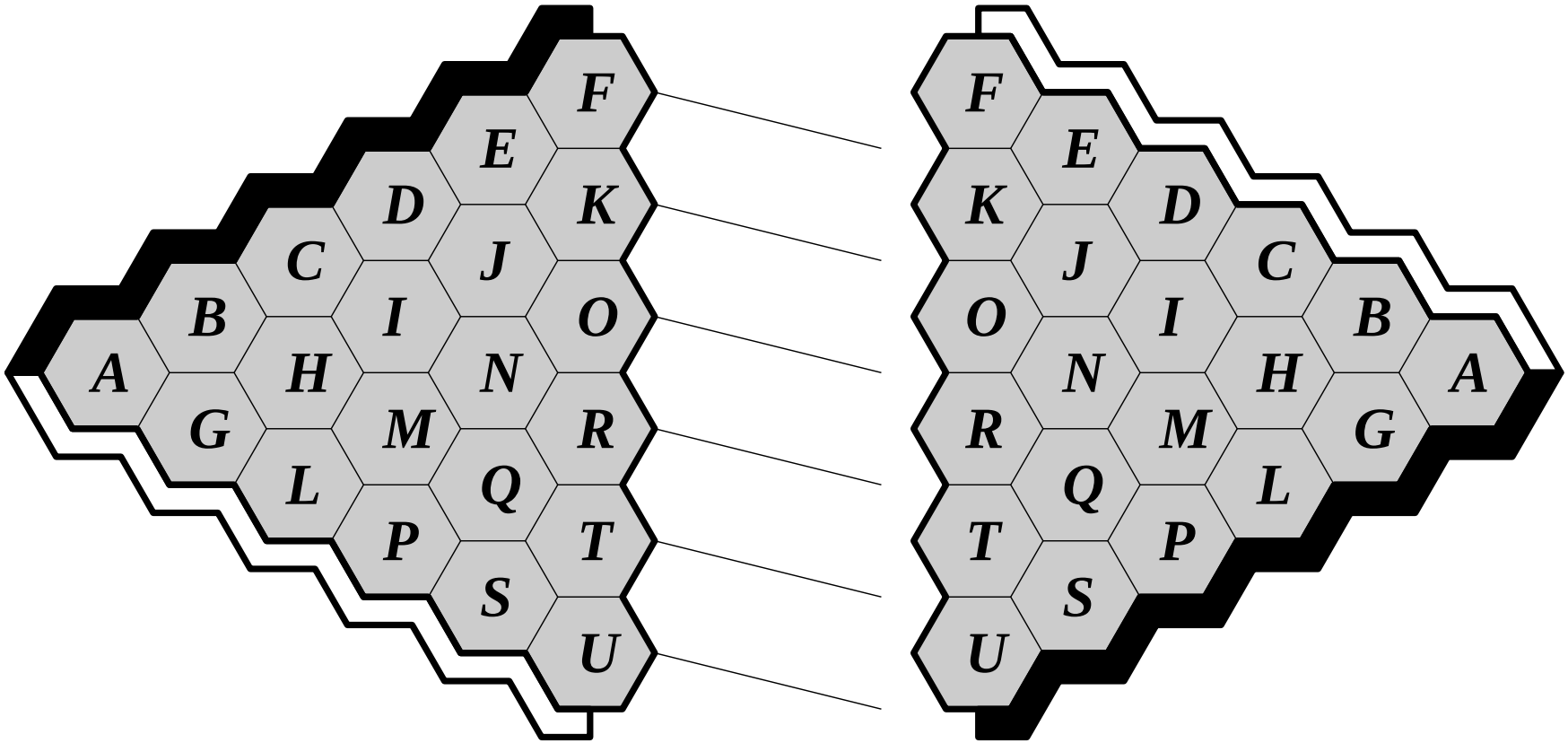
JOINS-CLOSER-EDGES STRATEGY



JOINS-CLOSER-EDGES STRATEGY



JOINS-CLOSER-EDGES STRATEGY



classic hex problems: strategies

CLASSIC HEX PROBLEMS: STRATEGIES

for 10×10 board:

find win/loss value of all 50 opening moves

so far: wins b9,e6 (on main diagonal), losses a1,a2

CLASSIC HEX PROBLEMS: STRATEGIES

for $n \times n$ boards with $n \geq 11$:

find a 1st-player winning strategy ?

find a winning 1st move ?

1st-move: short-diagonal-centermost always wins ?

dark Hex, a.k.a. Kriegspiel Hex

DARK HEX

- each player sees only their stones
- on a turn, private conversation with referee:

  can I move here?

  if yes: that is your move

  if no (opponent stone there): try again . . .

DARK HEX

- 3×3 center opening:

  has wins-with-probability-1 strategy

- 3×3 other openings: minimax strategy known

  (maximize your minimum expected win-rate,

  over all possible opponent strategies)

rex, cylindrical hex, Y

HEX VARIANTS

- rex (reverse hex)

  to lose? join your sides

- cylindrical hex (play on soup can label)

  to win? encircle or join top-bottom

- Y (3-sided board)

  to win? join all 3 sides

clobber

OTHER GAMES

- 2-player game, Nowakowski et al.
- black/white checkers on a checkerboard
- on a turn, clobber an adjacent opponent checker (you move, they leave)
- to win: make the last move

clobber problems

- linear clobber, starting config oxoxox. . . ox,

  prove 1st player wins

boxoff

OTHER GAMES

- 1-player game, Steven Meyers
- colored stones on a checkerboard (rectilinear)
- on a turn, remove any 2 same-colored stones that are the opposite corners of a rectangle (or opposite ends of a line segment) containing no other stones
- to win, remove all stones
- 5-colors (or more) boxoff is NP-complete

boxoff problems

- 2-color boxoff: in P?

thank you

Michael Johanson, Yngvi Björnsson, Morgan Kan, Nathan Po, Jack van Rijswijck, Broderick Arneson, Philip Henderson, Jakub Pawlewicz, Aja Huang (AlphaGo), Kenny Young, Noah Weninger, Chao Gao, Martin Müller, Bjarne Toft, Bedir Tapkan, Md-Reza Daliri, Peter Selinger, Eric Demer, Stephen Kennedy, Bob Hearn, Nancy Blachman

thank you

questions ?

email hayward@ualberta.ca