Well-posedness and dynamics of solutions to the generalized KdV with low power nonlinearity

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BIRS Workshop: New trends in Mathematics of Dispersive, Integrable and Nonintegrable Models in Fluids, Waves and Quantum Physics

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Outline

1. Introduction
2. Local well-posedness results
3. Numerical Investigations
4. Bibliography

Plan

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Introduction

In this talk, we study two types of the generalized Korteweg-de Vries equation: one, **gKdV**

$$\begin{cases} \partial_t u + \partial_x^3 u + u^{\alpha} \partial_x u = 0, & x, t \in \mathbb{R}, \\ u(x,0) = u_0, \end{cases} \tag{gKdV}$$

where the power $\alpha = \frac{m}{k}$ with $m, k \geq 1$ odd integers.

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where the power $\alpha = \frac{m}{k}$ with $m, k \geq 1$ odd integers.

We also consider the **GKdV**, with the absolute value incorporated into the nonlinearity

$$\begin{cases} \partial_t v + \partial_x^3 v + |v|^{\alpha} \partial_x v = 0, & x, t \in \mathbb{R}, \\ v(x,0) = v_0, \end{cases} \tag{GKdV}$$

where $\alpha > 0$.

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- **Local behavior**. We present local well-posedness (LWP) results for a subclass of $H^1(\mathbb{R})$. (By LWP we mean, existence, uniqueness and continuous dependence of the map data-to-solution).
- **Global behavior**. We use numerical methods to study the large time behavior of solutions. In this part, more differences between the two equations will be pointed out.

Introduction

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- The integer cases $k \geq 2$ have been used in several physical contexts, such as shallow-water waves among many others.
- The modular power nonlinearity as in GKdV ($|v|^\alpha \partial_x v$) has also been used in physics; for example, when $\alpha \in (0,1)$ it is studied in models of non-Maxwellian trapped electrons and description of their dynamics in ion-acoustic solitary waves

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- In the case $0 < \alpha < 1$, by using weighted spaces the well-posednesss was studied by Linares, Miyazaki and Ponce, 2019.
- In this talk, we show extensions of the previous well-posedness results to a wider class of fractional weights and $\alpha > 0$.

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- Formally, solutions of the gKdV and GKdV equations satisfy the mass and L^1-type conservation laws:

$$M[t] = \int \big(u(x,t)\big)^2 \, dx = M[0],$$

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- The energy is also conserved: in the gKdV case

$$E_{gKdV}[t] = \frac{1}{2}\int |\partial_x u(x,t)|^2 \, dx - \frac{1}{(\alpha+1)(\alpha+2)} \int \big(u(x,t)\big)^{\alpha+2} \, dx,$$

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and in the GKdV case

$$E_{GKdV}[t] = \frac{1}{2}\int |\partial_x v(x,t)|^2 \, dx - \frac{1}{(\alpha+1)(\alpha+2)} \int |v(x,t)|^{\alpha+2} \, dx$$

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- No other cases of gKdV or GKdV are known to be completely integrable.
- Both (gKdV) and (GKdV) equations are invariant under the scaling: if u solves one of them, then so does

$$u_\lambda(x,t) = \lambda^{\frac{2}{\alpha}} u(\lambda x, \lambda^3 t), \quad \lambda > 0.$$

Consequently, the (homogeneous) Sobolev space $\dot{H}^{s_c}$ is invariant under the scaling when

$$s_c = \frac{1}{2} - \frac{2}{\alpha}$$

.

Introduction

The traveling (solitary) wave solutions for both equations are of the form $u(x,t) = Q_c(x - ct - c_0)$, where $c > 0$ denotes the speed of propagation, c_0 is an initial shift, and Q_c is the rescaled ground state solution Q,

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$$Q_c(x) = c^{\frac{1}{\alpha}} Q(c^{\frac{1}{2}} x),$$

where Q is taken to be a smooth, positive, vanishing at infinity solution in the GKdV case of the equation:

$$-Q + Q'' + \frac{1}{(\alpha+1)} |Q|^{\alpha} Q = 0,$$

and in the gKdV case

$$-Q + Q'' + \frac{1}{(\alpha+1)} Q^{\alpha+1} = 0.$$

Introduction

Though technically the equations above are different, the positive (ground state) solutions are the same in both cases. In such case, we have

$$Q(x)=\left(\frac{(\alpha+1)(\alpha+2)}{2}\right)^{\frac{1}{\alpha}}\operatorname{sech}^{\frac{2}{\alpha}}\left(\frac{\alpha x}{2}\right),$$

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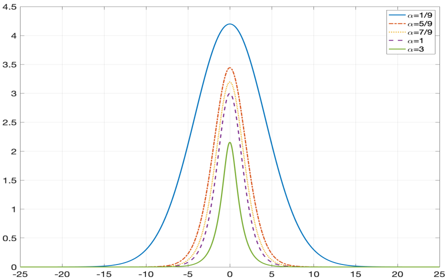

Figure 1: The ground state profiles Q for $\alpha=\frac{1}{9},\frac{5}{9},\frac{7}{9},1,3$.

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LWP results

Remark

One of the major difficulties for well-posedness is that the nonlinearities are not necessarily smooth

$$u^{\alpha}\partial_x u, \qquad |v|^{\alpha}\partial_x v$$

e.g., if $0 < \alpha < 1$, the function $z \mapsto |z|^{\alpha}$ is not of class C^1. *Classical methods of LWP are not expected to work in general.*

LWP results

Strategies

- The approach to obtain the existence is based on the work of Cazenave and Naumkin 2016, where authors developed a method to obtain local and global well-posedness for the NLS equation.

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- The approach to obtain the existence is based on the work of Cazenave and Naumkin 2016, where authors developed a method to obtain local and global well-posedness for the NLS equation.
- The idea is to consider initial conditions satisfying

$$\inf_{x\in\mathbb{R}} \langle x\rangle^{m}|u_0(x)| > 0,$$

where $\langle x\rangle^m = (1+|x|^2)^{\frac{m}{2}}$, and construct local solutions of the gKdV and GKdV equations from such data.

LWP results

Theorem (O. et al (2022))

Let $\alpha > 0$, $m \in \mathbb{R}^+$, $m > \max\{\frac{1}{2\alpha}, \frac{1}{2}\}$. *Let* $s \in \mathbb{Z}$ *with* $s \geq 2m+4$, *and assume that* u_0 *is a real-valued (or complex-valued) function such that*

$$v_0 \in H^s(\mathbb{R}),\ \langle x \rangle^m v_0 \in L^\infty(\mathbb{R}),\ \langle x \rangle^m \partial_x^j v_0 \in L^2(\mathbb{R}),\ j = 1,2,3,4, \tag{1}$$

$$\|v_0\|_{H^s} + \|\langle x \rangle^m v_0\|_{L^\infty} + \sum_{j=1}^{4} \|\langle x \rangle^m \partial_x^j v_0\|_{L^2} < \delta \tag{2}$$

for some $\delta > 0$ *and*

$$\inf_{x \in \mathbb{R}} \langle x \rangle^m |v_0(x)| =: \lambda > 0. \tag{3}$$

LWP results

Then there exist $T = T(\alpha, \delta, s, \lambda) > 0$ and a unique solution v of the GKdV equation (or a unique solution u of the gKdV equation with $\alpha = \frac{m}{k} > 0$, where m, k are odd integers) in the class

$$v \in C([0,T]; H^s(\mathbb{R})), \qquad \langle x \rangle^m \partial_x^j v \in C([0,T]; L^2(\mathbb{R})),\ j = 1,2,3,4 \tag{4}$$

with

$$\langle x \rangle^m v \in C([0,T]; L^\infty(\mathbb{R})), \qquad \partial_x^{s+1} v \in L^\infty(\mathbb{R}; L^2([0,T])), \tag{5}$$

and

$$\sup_{0 \le t \le T} \|\langle x \rangle^m (v(t) - u_0)\|_{L^\infty} \le \frac{\lambda}{2}. \tag{6}$$

Moreover, the map $u_0 \mapsto v(\cdot, t)$ is continuous in the following sense: for any compact $I \subset [0,T]$, there exists a neighborhood V of u_0 satisfying (1) and (3) such that the map is Lipschitz continuous from V into the class defined by (4) and (5).

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A key argument is the deduction of the following lemma, which relates the action of fractional weights on solutions of the linear KdV equation.

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A key argument is the deduction of the following lemma, which relates the action of fractional weights on solutions of the linear KdV equation.

Lemma

Let $m \in \mathbb{R}^+$. Then for any $t \in \mathbb{R}$, there exists $C > 0$ such that

$$\|\langle x\rangle^m e^{t\partial_x^3} f\|_{L^2} \leq C\langle t\rangle^{\lfloor m\rfloor+1}\big(\|J^{2m} f\|_{L^2} + \|\langle x\rangle^m f\|_{L^2}\big).$$

LWP results

Here, we denote by $\{e^{t\partial_x^3}\}_{t\in\mathbb{R}}$ the unitary group associated to solutions of the Airy equation $\partial_t u + \partial_x^3 u = 0$ with initial condition u_0.

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Idea proof decay lemma

We write $m = m_1 + m_2$, where $m_1 \in \mathbb{Z}^+ \cup \{0\}$, $m_2 \in [0,1)$. Then by Plancherel's identity and Leibniz's rule, we deduce

$$\begin{aligned}\|\langle x\rangle^m e^{t\partial_x^3} f\|_{L^2} \leq & C\|e^{t\partial_x^3} f\|_{L^2} + C\||x|^{m_2}|x|^{m_1} U(t) f\|_{L^2} \\ \leq & C\|f\|_{L^2} + C\|D^{m_2}\big(\frac{d^{m_1}}{d\xi^{m_1}}(e^{it\xi^3}\widehat{f}\,)\big)\|_{L^2}.\end{aligned}$$

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To deal with the local derivative, we use the identity

$$\frac{d^k}{d\xi^k}(e^{it\xi^3}) = e^{it\xi^3}\sum_{l=0}^{\lfloor\frac{2k}{3}\rfloor} c_l t^{k-l}\xi^{2k-3l},$$

LWP results

For the fractional part, we use one Stein's derivatives

$$\mathcal{D}^{\beta} f(x) = \left(\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^2}{|x-y|^{N+2\beta}} \, dy \right)^{1/2}, \; x \in \mathbb{R}^N.$$

Which satisfies $\|\mathcal{D}^{\beta} f\|_{L^2} = \|D^{\beta} f\|_{L^2} = \||\xi|^{\beta} \widehat{f}\|_{L^2}$.

LWP Results

Another key ingredient to study the nonlinear equation is the following sharp version of Kato's smoothing effect.

Lemma (Kenig-Ponce-Vega 1993)

For all $f \in L^2(\mathbb{R})$ complex or real valued,

$$\|e^{t\partial_x^3} f\|_{L_t^\infty L_x^2} + \|\partial_x U(t) f\|_{L_x^\infty L_t^2} = (1 + \frac{1}{\sqrt{3}})\|f\|_{L^2}.$$

LWP results

Our LWP is obtained by using the contraction mapping principle based on the integral formulation of gKdV or GKdV acting on the following space

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\begin{aligned}
\mathcal{X}_T = \big\{ & u \in C([0,T]; H^s(\mathbb{R})) : \\
\|u\|_{\mathcal{X}_T} := & \|u\|_{L^\infty_T H^s_x} + \|\langle x\rangle^m u\|_{L^\infty_T L^\infty_x} + \sum_{l=1}^{4} \|\langle x\rangle^m \partial_x^l u\|_{L^\infty_T L^2_x} \\
& + \|\partial_x^{s+1} u\|_{L^\infty_x L^2_T} \leq 2C_1\delta, \\
& \sup_{0\leq t\leq T} \|\langle x\rangle^m (u(t)-u(0))\|_{L^\infty} \leq \frac{\lambda}{2} \big\}.
\end{aligned}
$$

LWP results

Remarks

- An example of initial data that satisfies the conditions in the LWP theorem
$$u_0(x) = \frac{2\lambda e^{i\theta}}{\langle x\rangle^m} + \varphi(x), \quad \lambda \in \mathbb{R}, \quad \theta \in \mathbb{R},$$
with $\varphi \in \mathcal{S}(\mathbb{R})$ (the Schwartz class of functions).

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- Numerically we study solutions to the Cauchy problems gKdV and GKdV with initial data decaying at infinity as slow as $1/|x|$. We have LWP for a wider class of conditions with $\beta > \max\{\frac{1}{2\alpha}, \frac{1}{2}\}$ (for $\alpha > \frac{1}{2}$).

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- The class of initial data does not include any exponentially decaying data. for example, the ground state.

LWP results

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- Using the numerical approach, we are able to investigate the behavior of solutions which decay exponentially.

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- Using the numerical approach, we are able to investigate the behavior of solutions which decay exponentially.
- An interesting problem it to investigate analytically LWP for solutions of gKdV and GKdV in spaces that include the ground state Q (i.e., exponential spaces).

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Large time behavior

Soliton resolution conjecture

It formally states that a solution will eventually evolve into a finite number of solitons plus radiation, i.e.,

$$u(x,t) \approx \sum_{j=0}^{N} Q_{c_j}(x - c_j t - a_j) + r(x,t)$$

as $t \to \infty$, where $r(x,t)$ is the radiation and Q_c is some rescaled version of a soliton with a shift a_j and speed $c_j = c_j(t) \to c_j^*$.

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Numerical confirmation for gKdV and GKdV

We confirm the soliton resolution in various settings for single peak initial data (e.g., perturbations of solitons, Gaussian, super-Gaussian and polynomially decaying data).

Large time behavior

We consider Schamel's equation (GKdV with $|u|^{\frac{1}{2}}\partial_x u$) with initial condition $u_0(x) = Ae^{-x^2}$.

Remark

As typical for the KdV-type equations, a part of the solution propagates to the right as a soliton (or several solitons) and another part of the solution produces dispersive oscillations to the left, referred to as the radiation, decaying toward negative infinity

Large time behavior

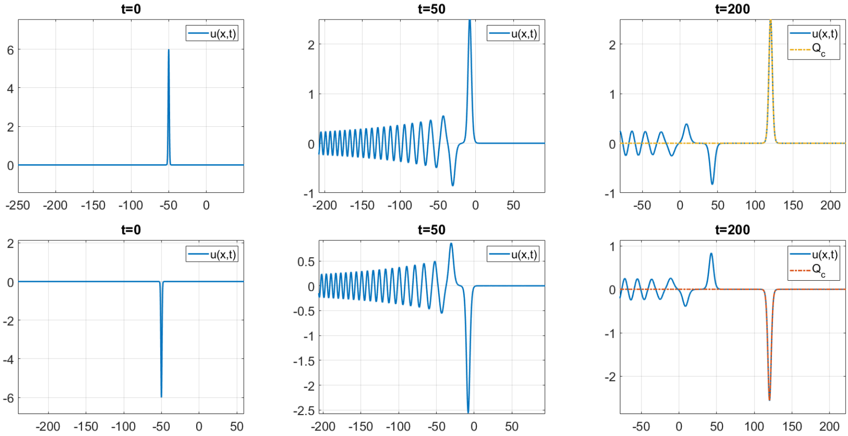

Figure 2: Time evolution in Schamel's equation of Gaussian data $u_0 = A\,e^{-x^2}$ (left), at $t = 50$ (middle) and $t = 200$ (right) with the fitting to the rescaled soliton Q_c. Top row: $A = 6$. Bottom row: $A = -6$.

Large time behavior

We consider initial condition

$$u_0(x) = v_0(x) = AQ(x + a)$$

We check the evolution of solutions of gKdV ($u^\alpha \partial_x u$), and GKdV ($|u|^\alpha \partial_x u$) when $A > 0$, $A < 0$.

Large time behavior: Case $A > 0$

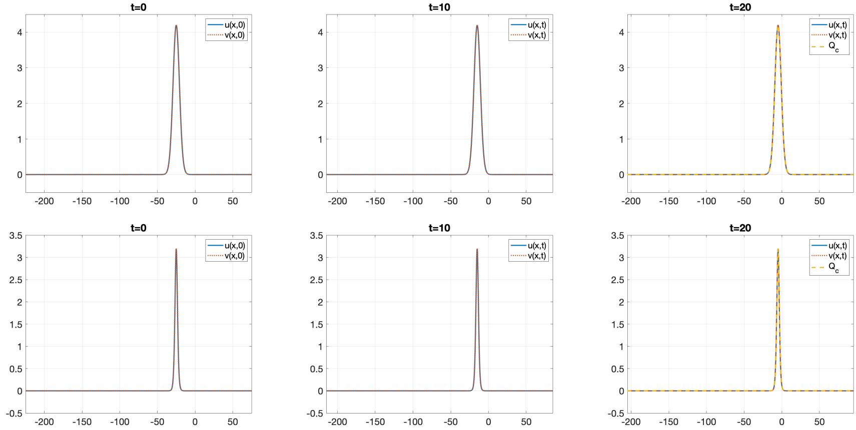

Figure 3: Time evolution for $u_0 = v_0 = Q(x+25)$ for $\alpha = \frac{1}{9}$ (top row) and $\alpha = \frac{7}{9}$ (bottom row); solution u of (gKdV) (solid blue) and v of (GKdV) (dotted red). Right column: both solutions are fitted with shifted $Q_c = Q$ from $(c = 1)$.

Large time behavior: Case $A < 0$

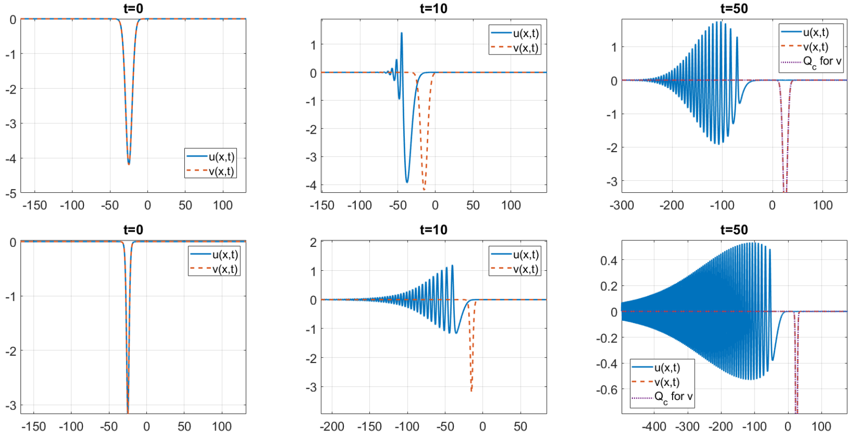

Figure 4: Time evolution for $u_0 = v_0 = -Q(x+25)$ for $\alpha = \frac{1}{9}$ (top row) and $\alpha = \frac{7}{9}$ (bottom row). Right column: the GKdV solution $v(x,t)$ (dashed red) fitted to shifted Q_c (dotted magenta).

Large time behaviour

We consider initial condition

$$u_0(x) = v_0(x) = Ae^{-(x-a)^2}.$$

We check the evolution of solutions of gKdV ($u^{\alpha}\partial_x u$), and GKdV ($|u|^{\alpha}\partial_x u$).

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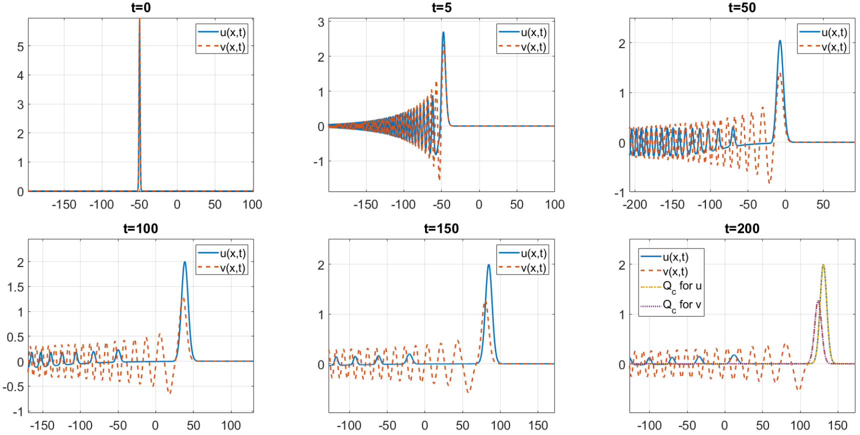

Figure 5: Time evolution for $u_0 = v_0 = Ae^{-(x+50)^2}$, $A = 6$ and $\alpha = \frac{1}{9}$.

Large time behavior: $A > 0$

Remark

- We observe what Miura called the "parade of solitons", i.e., the formation of a train of solitons with decreasing heights (or speed), and thus, eventually separating further and further from each other.

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- When $0 < \alpha \ll 1$, we don't need a large domain to observe the train of solitons, which is an advantage compared with integer powers.
- The solitons of gKdV ($u^\alpha \partial_x u$) model are slightly higher (and thus, faster) than the ones generated by the same data in the GKdV ($|v|^\alpha \partial_x v$) model.

Large time behavior: $A < 0$

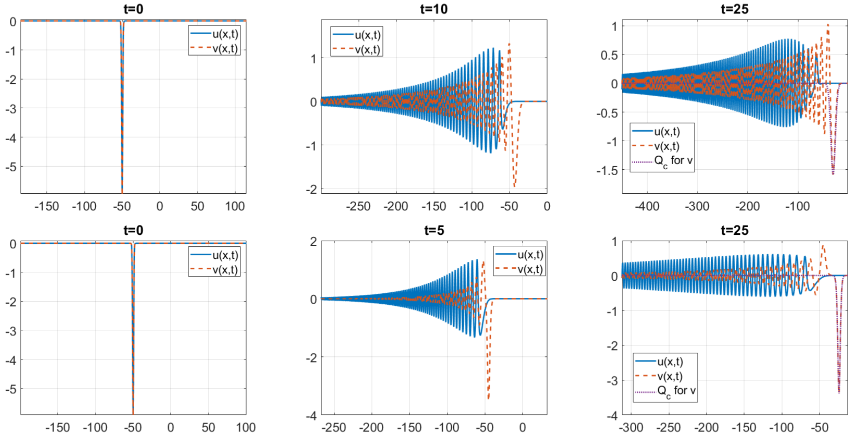

Figure 6: Time evolution for $u_0 = v_0 = A\,e^{-(x+a)^2}$. Top row: $\alpha = \frac{1}{9}$, $A = -6, a = 50$. Second row: $\alpha = \frac{7}{9}$, $A = -6, a = 50$.

Large time behavior

Remarks

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- Whereas the solutions to the GKdV evolve the negative bump into a (negative) soliton propagating to the right, and smaller in amplitude radiation outgoing to the left.
- The larger the power α is, the faster the separation of the soliton(s) from radiation occurs.
- We observe that in GKdV the formation of solitons is not influenced by the sign of the initial condition.

Large time behaviour

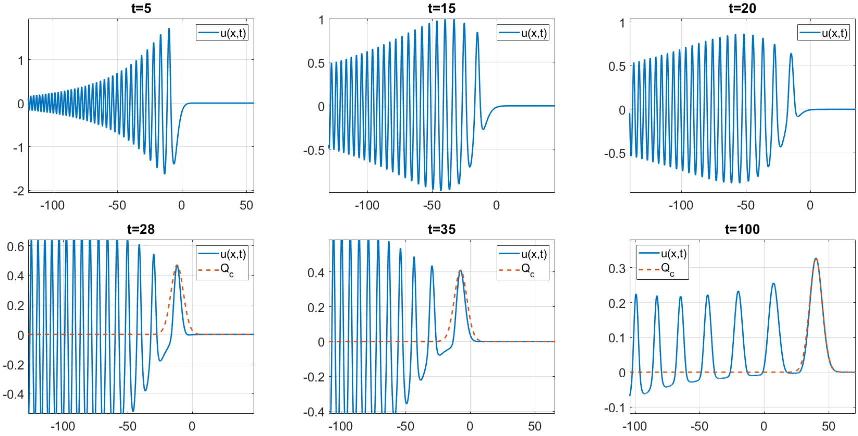

Figure 7: The gKdV time evolution for $u_0 = -6\, e^{-x^2}$, $\alpha = \frac{1}{9}$, till $t = 100$.

Large time behavior

Remarks

- One can observe that the first negative bump in the radiation decreases in its magnitude (becomes smaller, see the top row), and then eventually disappears.

Large time behavior

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- One can observe that the first negative bump in the radiation decreases in its magnitude (becomes smaller, see the top row), and then eventually disappears.
- Then the next positive bump starts to separate from the pulse-like radiation, and because it is positive (in gKdV model), it starts forming a soliton, or asymptotically approaches a rescaled version of it.

Large time behavior

Remarks

- We also study the cases $\alpha \to 0$. We find that the smaller the power α is, the longer the time the solution evolves into a rescaled soliton (the biggest bump) and the shorter the height of that bump is (for the same data).

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- We *conjecture* that when $\alpha \to 0$, the time the main lump evolves into a soliton will go to ∞ as well.
- We also study the iteration of two bump profiles.

Plan

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